Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 144 (2009) 1119–1134
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

When is there state independence?

Brian Hill *

GREGHEC, HEC School of Management, Paris, 1 rue de la Libération, 78351 Jouy-en-Josas, France

Received 22 November 2007; final version received 8 October 2008; accepted 30 November 2008

Available online 25 December 2008

Abstract

Whether a preference relation can be represented using state-independent utilities as opposed to state-dependent utilities may depend on which acts count as constant acts. This observation underlies an extension of Savage's expected utility theory to the state-dependent case that was proposed in this journal by Edi Karni. His result contains a condition requiring the existence of a set of acts which can play the role of constant acts and support a representation involving a state-independent utility function. This paper contains necessary and sufficient conditions on the preference relation for such a set of acts to exist. Results are obtained both for the Savage and the Anscombe and Aumann frameworks. Among the corollaries are representation theorems for state-dependent utilities. Relationships to Karni's work and extensions of the results are discussed.

JEL classification: D81; C69

Keywords: Subjective expected utility; State-dependent utility; Monotonicity axiom

1. Introduction

There has been much discussion of representations of preferences with state-dependent utilities [3,7–11,13]. In several of these discussions, it has been noted that, by redefining which acts (i.e. functions from states to consequences) count as constant, one can transform a state-independent representation into a state-dependent one, and conversely, a state-dependent representation into a state-independent one. To take the example proposed in [13], if the acts are bets on the exchange rates between dollars and yen, then a representation which is state-independent

* Fax: +33 1 39 67 71 09.
E-mail address: hill@hec.fr.

0022-0531/$ – see front matter
doi:10.1016/j.jet.2008.11.008

when the stakes are formulated in one currency will be state-dependent when the stakes are formulated in the other. Indeed, the idea that failures of Savage's state-independence axioms may come about because the consequences do not yield the acts which are "really" constant is behind some theories of state-dependent utility. Most notably, the extension of Savage's expected utility theory [12] proposed several years ago by Edi Karni in this journal [8], and the extension he proposed of Anscombe and Aumann's version of this theory [1,7] rely precisely on this idea. In particular, he introduces the notion of *constant valuation acts*, which, although they are not constant acts, play the role of constant acts in Savage- or Anscombe-and-Aumann-like representation theorems: state-independence holds with respect to the constant valuation acts instead of the constant acts.

Under what conditions does there exist a set of acts that can play the role of constant acts and that yield a state-independent utility representation of this sort? In [8], Karni simply poses the existence of such a set of acts as a condition in his theorem. However, standard economic methodology prefers non-technical conditions appearing in representation theorems to be formulated directly in terms of the preference relation. This paper provides conditions of this sort: that is, properties of the preference relation which are necessary and sufficient for the existence of a set of acts which can play the role of constant acts and support a state-independent utility representation. Results will be presented both for the Savage framework and the Anscombe and Aumann framework.

To be more precise, state independence comes in two flavours: *ordinal* state independence, according to which the preference order on constant acts is independent of the state, and the stronger notion of *cardinal* state independence, according to which the numerical utilities assigned to the consequences of constant acts are independent of the state. So the initial question can be understood in two ways, depending on which concept of state-dependence is of interest. This paper is mainly concerned with conditions for the existence of a set of acts that can play the role of constant acts and with respect to which there is ordinal state independence; however, the results will yield as a corollary a further condition which, in a large class of cases, guarantees that there is cardinal state independence with respect to this set of acts. Monotonicity – the traditional axiom for ordinal state independence – states that, for any pair of constant acts, the first is preferred to the second if and only if, for any non-null event, the first is preferred to the second given that event (Definition 2.3). We say that *essential monotonicity* holds if there is a set of acts which can play the role of constant acts and which satisfies monotonicity (Definition 2.4). The main result of this paper will be conditions on the preference relation that are necessary and sufficient for essential monotonicity to hold.

As an illustration, consider the tables in Fig. 1, each representing a simple decision problem where there are three outcomes – c_1, c_2, c_3 – and two states – s_1, s_2; the entries in the respective tables indicate the preference orders on the outcomes conditional on the states.[1] The preference relation displayed in the left-hand table is not state independent. Consider now the set containing the three acts f_1, f_2 and f_3, where $f_1(s_1) = c_1$, $f_1(s_2) = c_3$, $f_2(s_1) = c_2$, $f_2(s_2) = c_2$, $f_3(s_1) = c_3$ and $f_3(s_2) = c_1$. This set of acts does satisfy the monotonicity condition: they lie in the same relation according to the preference order conditional on s_1 as according to the preference relation conditional on s_2. Furthermore, this set of acts can play the role of constant acts, because it has the following property (Section 2.2): for each state and each outcome, there is a

[1] We are assuming, for the sake of this example, the basic postulates of Savagean decision theory (weak order and Savage's sure-thing principle, or Anscombe and Aumann's independence), under which there is a well-defined concept of preference order conditional on a state.

	s_1	s_2
c_3	first	third
c_2	second	second
c_1	third	first

<table>
<tr><th></th><th>s_1</th><th>s_3</th></tr>
<tr><td>c_3</td><td>first</td><td>first</td></tr>
<tr><td>c_2</td><td>second</td><td rowspan="2">second equal</td></tr>
<tr><td>c_1</td><td>third</td></tr>
</table>

Fig. 1. Essential state independence.

unique act in the set which yields that outcome on that state. Hence, although monotonicity does not hold in this case, essential monotonicity does: there is a set of acts which can play the role of constant acts and which satisfies the monotonicity condition.

By contrast, essential monotonicity does not hold in the example the right-hand table. There is no set of acts that has the property mentioned above and satisfies monotonicity. Such a set of acts must take different values on each state. So, for each such set, there is a strict preference between any pair of acts in the set according to the preference order conditional on s_1. However, there exists a pair of acts of the set between which the preference order conditional on s_3 is indifferent (those which take the values c_1 and c_2). Hence, for each set of acts which can play the role of constant acts, there are pairs of acts which lie in different relations according to the preference orders conditional on s_1 and conditional on s_3. Essential monotonicity does not hold.

The problem in this case seems to be that the preference orders conditional on the different states do not have sufficient structure in common for there to exist a set of acts that could play the role of constant acts and satisfy the monotonicity property. The main results of this paper exploit this observation. Monotonicity implies that, to support an ordinally state-independent representation, a set of acts which may count as constant should be such that one of these acts is preferred to another given a particular state if and only if the former is preferred to the latter given any other state. This yields the condition that, for any two states, the order on the outcomes given one state is isomorphic to the order on the outcomes given the other.

This basic idea needs to be qualified. For one, the reasoning only applies to non-null states. This qualification is all that one needs to obtain necessary and sufficient conditions for essential monotonicity when working in the Anscombe and Aumann framework (Theorem 3.1, Section 3). However, in the Savage framework, where all states are null, a further technical condition is required; once again, a set of necessary and sufficient conditions for essential monotonicity can be obtained (Theorem 4.1, Section 4).

A first corollary of the main results are representation theorems, comparable to the one in [8]. The condition requiring the existence of a set of acts which satisfies monotonicity is replaced by a condition requiring that the conditional preference orders are isomorphic. One thus obtains new representation theorems, in both the Anscombe and Aumann and Savage frameworks (Theorems 3.2 and 4.2).

As a second corollary, the main results imply, in a large class of cases, a strong uniqueness property on the sets of acts which satisfies monotonicity and can play the role of constant acts. This yields, in such cases, necessary and sufficient conditions for there to be a set of acts with respect to which there is cardinal state independence.

Section 2 introduces the decision-theoretic frameworks and the basic notions. Necessary and sufficient conditions for essential monotonicity will be then given in the Anscombe and Aumann framework (Section 3) and in the Savage framework (Section 4). The corollaries mentioned above will be drawn. Section 5 discusses relationship with previous work, notably by Karni [7,8], and possible extensions of the results. Proofs of the main results are in Appendix A.

2. Preliminaries and basic concepts

2.1. The two frameworks

The basic notions and results of this paper apply both in the Anscombe and Aumann framework and in the Savage framework. In both cases, S will designate the set of *states of nature* and C the set of *outcomes*.

In the Anscombe and Aumann framework [1], S is a finite non-empty set. Subsets of S are called *events*. C is a non-empty finite or countably infinite set; a *consequence* is a probability measure on C with finite support. $\Delta(C)$ is the set of consequences. The set of consequences which assign weight 1 to one outcome is in one-to-one correspondence with the set of outcomes; we denote the former set also by C, with slight abuse of notation. Let H denote the set of mappings $h : S \to \Delta(C)$. In this framework, there is a mixture structure on H, i.e. for $h, h' \in H$ and $\alpha \in [0, 1]$, there is $\alpha h + (1 - \alpha)h' \in H$, which is defined by $(\alpha h + (1 - \alpha)h')(s) = \alpha h(s) + (1 - \alpha)h'(s)$.[2] We call the elements of H taking values in C *acts*, to allow simpler comparison with the Savage framework. $\mathcal{A}$ is the set of acts. A preference relation $\preccurlyeq$ is assumed on H; we shall refer to its restriction to $\mathcal{A}$ also by $\preccurlyeq$.

In the Savage framework [12], S is an infinite set which is equipped with a σ-algebra $\mathscr{S}$, such that, for any $s \in S$, $\{s\} \in \mathscr{S}$. The elements of $\mathscr{S}$ are called *events*. C is a non-empty finite or countably infinite set. Here, C is the set of *consequences*. An *act* is a function $f : S \to C$ which is measurable with respect to $(S, \mathscr{S})$. $\mathcal{A}$ is the set of acts; a preference relation $\preccurlyeq$ is assumed on it.

For the remainder of Section 2, everything said will apply to both frameworks.

Throughout this paper, f, g, and so on will designate members of $\mathcal{A}$. As usual, $f \sim g$ expresses indifference according to the preference order, and, for any event A, $\preccurlyeq_A$ denotes the preference order given A.[3] We write $f \preccurlyeq_s g$ instead of $f \preccurlyeq_{\{s\}} g$. For any consequence $c \in C$ (respectively $p \in \Delta(C)$), c will denote the element of $\mathcal{A}$ (respectively H) taking value c (respectively p) for all $s \in S$.

Definition 2.1. An event A is *null* if, for any acts f, g such that $f(s) = g(s)$ for any $s \notin A$, $f \sim g$. A state s is said to be null if the event $\{s\}$ is null.

2.2. Basis

We wish to consider a generalised monotonicity property involving not constant acts but another set of acts. The set of constant acts has several important properties that any set which can be used instead of it in a generalised monotonicity property should also possess. In particular, for any state and any outcome, there is a unique constant act which, applied to the state, yields the outcome. We call any set of acts with this property a basis.

Definition 2.2 *(Basis).* A *basis* $\mathcal{B}$ is a set of $b^i \in \mathcal{A}$, such that, for each $s \in S$ and for each $c \in C$, there exists $b^i \in \mathcal{B}$ with $b^i(s) = c$ and for all $j \neq i$, $b^j(s) \neq c$.

---

[2] Anscombe and Aumann's Reversal of Order axiom [1, p. 201] is assumed here.

[3] For all $f, g \in \mathcal{A}$, $f \preccurlyeq_A g$ iff, for any $f', g' \in \mathcal{A}$ with $f'(s) = f(s)$ and $g'(s) = g(s)$ for any $s \in A$ and $f'(s) = g'(s)$ for $s \notin A$, $f' \preccurlyeq g'$. Since we assume independence and the sure-thing principle (see Sections 3 and 4), this coincides with the ordinary notion of conditional preferences.

Let $\mathcal{B}_C$ be the basis of constant acts.

Bases span, in a unique way, the set of all acts: an act, as a measurable function from states to outcomes, can equally be thought of as a measurable function from states to elements of the basis. This is the content of the following proposition.

Proposition 2.1. *Consider a basis $\mathcal{B}$. For each $f \in \mathcal{A}$, there is a unique measurable function $f^b : S \to \mathcal{B}$ such that $f(s) = f^b(s)(s)$.*

Different bases afford different ways of expressing the same acts, just as different bases of a vector space afford different ways of referring to the same vector.

An alternative way of understanding the notion of basis is as follows. For any pair of states s and s', the basis $\mathcal{B}$ associates to any outcome c a unique outcome d such that obtaining c on s "corresponds" to obtaining d on s'. So, for any s and s', $\mathcal{B}$ generates a bijective (i.e. one-to-one and onto) mapping $\psi_{ss'} : C \to C$. The family of mappings $\{\psi_{ss'}\colon\ s, s' \in S\}$ generated by $\mathcal{B}$ is closed under composition (for any $s, s', s'' \in S$, $\psi_{s's''} \circ \psi_{ss'} = \psi_{ss''}$) and contains the identities (ψ_{ss} is the identity, for each $s \in S$). The converse also holds: for any family Σ of bijective functions from C to itself, indexed by pairs of elements of S, that is closed under composition and contains the identities, there corresponds a unique basis. Formally:

Proposition 2.2. *Given a basis $\mathcal{B}$, there is a unique family Σ of bijective functions $\psi_{ss'} : C \to C$ for each $s, s' \in S$ which contains the identities and is closed under composition, such that, for each $s, s' \in S$ and every $b^i \in \mathcal{B}$, $\psi_{ss'}(b^i(s)) = b^i(s')$. Conversely, given such a family Σ of bijective functions containing the identities and closed under composition, there exists a unique basis $\mathcal{B}$ such that, for every pair $s, s' \in S$ and every $b^i \in \mathcal{B}$, $\psi_{ss'}(b^i(s)) = b^i(s')$.*

Thus, instead of thinking in terms of bases, one can equivalently think in terms of families of functions which identify which outcome given one state corresponds to a particular outcome given another state. Whilst the former perspective is adopted in this paper, [7,8] use the latter.[4]

2.3. Monotonicity

As stated in the Introduction, the axiom with which we are mainly concerned here is monotonicity. Here is a formulation common to the Savage and Anscombe and Aumann frameworks (it is visibly identical to Savage's P3; the relation to Anscombe and Aumann's monotonicity in prizes axiom will be discussed in Section 3.2).

Definition 2.3. *Monotonicity* holds if for every non-null event A, and for all constant acts c and d, $c \preccurlyeq_A d$ iff $c \preccurlyeq d$.

Monotonicity refers to the basis of constant acts. The idea that the constant acts may not be "really" constant, but that another set of acts may be instead, leads to the following weakening of the axiom:

[4] Notwithstanding the remark on [7, pp. 192–193].

Definition 2.4. *Essential monotonicity* holds if there exists a basis $\mathcal{B}$ such that, for every non-null event A, and for all $b^i, b^j \in \mathcal{B}$, $b^i \preccurlyeq_A b^j$ iff $b^i \preccurlyeq b^j$. Such a basis is an *essentially monotonic basis*.

Just as monotonicity is an ingredient in traditional representation theorems, essential monotonicity can be used in generalised representation theorems (Sections 3.2 and 4.2). However, by contrast with many standard axioms of decision theory, essential monotonicity, as stated, is not an immediate condition on the preference relation. The next section presents a condition formulated directly in terms of the preference relation which is equivalent to essential monotonicity in the Anscombe and Aumann framework and the following section does the same for the Savage framework.

3. Characterising essential monotonicity: the Anscombe and Aumann framework

3.1. Background assumptions

From the Anscombe and Aumann axioms other than those concerning state-independence (i.e. weak order, independence, continuity and reversal of order, but not monotonicity in prizes), it follows that there is a representation of $\preccurlyeq$ by a function $U : S \times C \to \Re$ such that, for any $h, h' \in H$, $h \preccurlyeq h'$ iff

$$\sum_{s\in S} \sum_{c\in \mathrm{supp}(h(s))} U(s,c)h(s)(c) \leqslant \sum_{s\in S} \sum_{c\in \mathrm{supp}(h(s))} U(s,c)h(s)(c). \tag{3.1}$$

Further, this function is unique up to similar positive affine transformations (or, to use the terminology in [11], cardinal unit comparable transformations): U' satisfies (3.1) if and only if there is a positive real number a and real numbers b_s for each $s \in S$ such that $U'(s,c) = aU(s,c) + b_s$ for all $s \in S$, $c \in C$ [4, Chapter 13].

Throughout this section, it is assumed that $\preccurlyeq$ satisfies conditions sufficient for a representation of the form (3.1).

3.2. Ordinal and cardinal state independence

Given the background assumptions, the following is equivalent to Anscombe and Aumann's axiom for state-independence of utility, monotonicity in prizes.

Definition 3.1. *Monotonicity in consequences* holds if for every non-null event A, and for all constant-valued $p, q \in H$, $p \preccurlyeq_A q$ iff $p \preccurlyeq q$.

Monotonicity, as defined in Definition 2.3, is monotonicity in outcomes (elements of C) rather than monotonicity in consequences (elements of H). Monotonicity in outcomes only ensures *ordinal* state independence, while the stronger monotonicity in consequences is required to ensure *cardinal* state independence.

We thus propose factorising the monotonicity-in-consequences condition into an ordinal state-independence condition – the notion of monotonicity in Definition 2.3 – and a condition (to be stated below) which, given ordinal state independence, yields cardinal state independence. Not only does this underline the common ground with the Savage framework, where there are two axioms for state independence (Section 4.2), but it also allows a more precise delineation of the subject of this paper. We are considering the weakening of ordinal state independence that can be obtained by choosing a new set of constant acts.

The condition for cardinal state independence must presuppose that the generalised condition for ordinal state independence is satisfied; namely, that there is an essentially monotonic basis. In the standard case this is assumed to be the set of constant acts, but in the interests of generality, no assumptions will be made regarding it here, so it has to be explicitly mentioned.

Definition 3.2. *Cardinal state independence* holds with respect to an essentially monotonic basis $\mathcal{B}$ if for any $h, h' \in H$ which are mixtures of elements of $\mathcal{B}$, i.e. $h = \sum_{i=1}^{n} \alpha_i b^i$ and $h' = \sum_{j=1}^{m} \beta_j b^j$ with $\alpha_i, \beta_j \geqslant 0$, $\sum \alpha_i = \sum \beta_j = 1$, and for any non-null event A, $h \preccurlyeq_A h'$ iff $h \preccurlyeq h'$.

Clearly, the conjunction of essential monotonicity with essentially monotonic basis $\mathcal{B}$ and cardinal state independence with respect to $\mathcal{B}$ is equivalent to a generalised version of monotonicity in consequences, where the implicit use of the basis of constant acts is replaced by the essentially monotonic basis $\mathcal{B}$. Anscombe and Aumann's representation theorem [1] can thus be immediately generalised, replacing monotonicity in consequences by the conjunction of essential monotonicity and cardinal state independence with respect to an essentially monotonic basis $\mathcal{B}$, and the representation by a probability and a utility function on the set of outcomes by a representation by a probability and a utility function on the basis $\mathcal{B}$. That is, there is a representation of the following form:

$$\sum_{s\in S} \sum_{\substack{b^i \in \mathcal{B} \text{ s.t.} \\ b^i(s)\in \operatorname{supp}(h(s))}} p(s)u\big(b^i\big)h(s)\big(b^i(s)\big) \leqslant \sum_{s\in S} \sum_{\substack{b^i \in \mathcal{B} \text{ s.t.} \\ b^i(s)\in \operatorname{supp}(h'(s))}} p(s)u\big(b^i\big)h'(s)\big(b^i(s)\big) \quad (3.2)$$

where p is unique, and u is unique up to positive affine transformations. This generalisation can be thought of as a representation theorem for state-dependent utility, insofar as Eq. (3.2) can be reformulated as

$$\sum_{s\in S} \sum_{\substack{c\in C \text{ s.t.} \\ c\in \operatorname{supp}(h(s))}} p(s)u'(s,c)h(s)(c) \leqslant \sum_{\substack{c\in C \text{ s.t.} \\ c\in \operatorname{supp}(h(s))}} p(s)u'(s,c)h'(s)(c)$$

with $u'(s,c) = u(b^i)$, where $b^i(s) = c$. This was one of the motivations of [7,8].

3.3. Characterising essential monotonicity

The aim is to find a property of the preference relation which is necessary and sufficient for the existence of an essentially monotonic basis. The major obstacle to the existence of this sort of basis was illustrated in the introduction: if, for non-null states s and t, $\preccurlyeq_s$ and $\preccurlyeq_t$ do not have a sufficient number of properties in common then no essentially monotonic basis exists. Intuitively, it seems evident how much the two orders need to have in common: they should be isomorphic. The main theorem of this section confirms that this is all that is required. First a preliminary definition.

Definition 3.3. Two ordered sets (or orders, for short) $(X, \preccurlyeq_X)$ and $(Y, \preccurlyeq_Y)$ are *isomorphic* if there exists a bijective function $\psi : X \to Y$ such that, for all $x, x' \in X$, $x \preccurlyeq_X x'$ iff $\psi(x) \preccurlyeq_Y \psi(x')$.

For any state s, let $(C, \preccurlyeq_s)$ be the set C equipped with $\preccurlyeq_s$. The condition that the preference orders conditional on states have the same structure only applies to non-null states.

Definition 3.4. *Isomorphic conditional preferences* holds if for any non-null $s_1, s_2 \in S$, $(C, \preccurlyeq_{s_1})$ and $(C, \preccurlyeq_{s_2})$ are isomorphic.

The necessity and sufficiency of this condition is expressed by the following theorem.

Theorem 3.1. *Essential monotonicity holds iff isomorphic conditional preferences holds.*

Taken in tandem with the generalisation of Anscombe and Aumann's theorem described above, Theorem 3.1 has the following representation theorem as an immediate corollary.

Theorem 3.2. *Suppose that there is a representation of* $\preccurlyeq$ *of sort specified in* (3.1) *which is unique up to similar positive affine transformations. Then*:

1. *The following are equivalent*:
   (i) *isomorphic conditional preferences holds and cardinal state independence holds with respect to an essentially monotonic basis* $\mathcal{B}$;
   (ii) *there exists a probability distribution* p *on* S, *a basis* $\mathcal{B}$ *and a real-valued function* u *on* $\mathcal{B}$, *such that, for* $h, h' \in H$, $h \preccurlyeq h'$ *iff*

$$\sum_{s\in S} \sum_{\substack{b^i \in \mathcal{B} \text{ s.t.} \\ b^i(s)\in \operatorname{supp}(h(s))}} p(s)u\big(b^i\big)h(s)\big(b^i(s)\big) \leqslant \sum_{s\in S} \sum_{\substack{b^i \in \mathcal{B} \text{ s.t.} \\ b^i(s)\in \operatorname{supp}(h'(s))}} p(s)u\big(b^i\big)h'(s)\big(b^i(s)\big). \tag{3.3}$$

2. *For a given* $\mathcal{B}$, p *is unique, and* u *is unique up to positive affine transformations.*

As noted above, this can be thought of as a representation theorem for state-dependent utility.

A corollary of the proof of Theorem 3.1 is the following uniqueness property for essentially monotonic bases.

Corollary 3.1. *Suppose that essential monotonicity holds, and let* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be two essentially monotonic bases. Suppose furthermore that, for any non-null* s, *there is either a minimal or a maximal element* (*or both*) *in* $(C, \preccurlyeq_s)$. *Then, for each* $b_1^i \in \mathcal{B}_1$, *there is a* $b_2^j \in \mathcal{B}_2$ *such that, for any non-null event* A, $b_1^i \sim_A b_2^j$. *Furthermore, for each* $b_1^i \in \mathcal{B}_1$, $|\{b_2^j \in \mathcal{B}_2 \mid$ *for any non-null event* A, $b_1^i \sim_A b_2^j\}| = |\{b_1^k \in \mathcal{B}_1 \mid b_1^i \sim b_1^k\}|$.

In a word, if $(C, \preccurlyeq_s)$ has a maximal or minimal element, the only differences between essentially monotonic bases occur when there are several outcomes on which $\preccurlyeq_s$ is indifferent. For example, there could be essentially monotonic bases $\mathcal{B}$ and $\widehat{\mathcal{B}}$, with the first containing elements b^i, b^j with $b^i(s_1) = c$, $b^i(s_2) = d$, $b^j(s_1) = c'$, and $b^j(s_2) = d'$ and the second containing elements $\widehat{b^i}$, $\widehat{b^j}$ with $\widehat{b^i}(s_1) = c$, $\widehat{b^i}(s_2) = d'$, $\widehat{b^j}(s_1) = c'$, and $\widehat{b^j}(s_2) = d$, but only if $c \sim_{s_1} c'$ and $d \sim_{s_2} d'$.

The requirement that there is a maximal or minimal element is necessary, as can be seen from the case where the preference orders conditional on states are isomorphic to the order on the integers $((\mathbb{Z}, \leqslant))$. Any basis obtained from a given essentially monotonic basis by "shifting" the values that the elements take on a particular state by a constant number of rungs is also an essentially monotonic basis.

Corollary 3.1 says that whenever C is finite or countably infinite with a maximal or minimal element according to $\preccurlyeq_s$, there is little degree of freedom in the choice of essentially monotonic basis, supposing that essential monotonicity holds. In such cases, there is thus no degree of freedom in whether cardinal state independence holds: cardinal state independence holds with respect to a particular essentially monotonic basis if and only if it holds with respect to *any* essentially monotonic basis. In such cases, Theorem 3.2 can be strengthened: "an essentially monotonic basis" in Clause 1(i) can be replaced with "any essentially monotonic basis." Furthermore, the relativity to the basis in the uniqueness clause (Clause 2) can be effectively removed, since the same probabilities and utilities (up to positive affine transformations) are obtained for all essentially monotonic bases. This is not so in cases of unbounded $\preccurlyeq_s$. Consider a development of the previous example where the consequences are the set of integers and the utility of a consequence c given any state s is $2.c$ if $c > 0$ and c if not. There is cardinal state independence with respect to the basis of constant acts, $\mathcal{B}_C$; however, for any basis obtained from $\mathcal{B}_C$ by "shifting" in the way described above, although it is essentially monotonic, cardinal state independence does not hold with respect to it.

4. Characterising essential monotonicity: the Savage framework

4.1. Background assumptions

We want to assume an equivalent of the representation (3.1) for the Savage framework. Since the set of states S is infinite, the double sum must be replaced by an integral over the product space $S \times C$. Accordingly the function U is replaced by a measure on $(S \times C, \mathscr{T})$, where the σ-algebra $\mathscr{T}$ is the product of $\mathscr{S}$ and the power set of C. Finally, an act $f \in \mathcal{A}$ is replaced by its graph $(\{(s, c) \mid f(s) = c\})$; since f is a measurable function, its graph is a measurable set in $(S \times C, \mathscr{T})$. (We use the symbol f to refer both to the function and its graph.) We assume here that $\preccurlyeq$ satisfies conditions sufficient for the existence of a measure U on $(S \times C, \mathscr{T})$ such that, for any $f, f' \in \mathcal{A}$, $f \preccurlyeq f'$ iff

$$\int_f dU \leqslant \int_{f'} dU \tag{4.1}$$

and such that this measure is unique up to similar positive affine transformations: U' satisfies (4.1) if and only if there exists $a > 0$ and a measurable function $b : S \to \Re$ such that $U' = aU + b$. Sufficient conditions for this representation have been proposed in Hill [6]: they correspond more or less to the basic Savage axioms except for those concerning state-independence (in particular, weak order, the sure-thing principle and continuity axioms, but not monotonicity and weak comparative probability).

4.2. Ordinal and cardinal state independence

As in the Anscombe and Aumann framework, monotonicity (Definition 2.3) only ensures *ordinal* state independence, but not *cardinal* state independence (see also [8,9]). The further condition proposed by Savage is his P4 (also known as weak comparative probability), which we shall call *cardinal state independence*, to underline the analogy with the Anscombe and Aumann framework. The traditional formulation assumes that the set of constant acts is an essentially monotonic basis; in the interests of generality, assumptions regarding the appropriate essentially monotonic basis are dropped in the formulation used here.

Definition 4.1. *Cardinal state independence* holds with respect to an essentially monotonic basis $\mathcal{B}$ if for every pair of events A and B and every $b^i, b^j, b^k, b^l \in \mathcal{B}$ such that $b^i \prec b^j$ and $b^k \prec b^l$, $b^i{}_A b^j \preccurlyeq b^i{}_B b^j$ iff $b^k{}_A b^l \preccurlyeq b^k{}_B b^l$, where $b^i{}_A b^j$ is the act which takes the values of b^i on A and the values of b^j on A^c.

Just as there is an immediate generalisation of Anscombe and Aumann's theorem (Section 3.2), there is an immediate generalisation of Savage's representation theorem [12], where his state-independence axioms are replaced by essential monotonicity and cardinal state independence with respect to an essentially monotonic basis $\mathcal{B}$, and these are necessary and sufficient, given the background assumptions, for a representation of the following form:

$$\int_S u\big(f^b(s)\big)\,dp \leqslant \int_S u\big(f'^b(s)\big)\,dp$$

where f^b and f'^b are as in Proposition 2.1 and p and u have the ordinary uniqueness properties. As for the Aumann and Anscombe case, this generalisation of Savage's theorem can be thought of as a representation theorem for state-dependent utility.

4.3. Characterising essential monotonicity

The isomorphic conditional preferences condition used in the Anscombe and Aumann framework (Definition 3.4) cannot be directly applied in the Savage framework, because in Savage's theory, where the state space is atomless, all states are null, so the condition is trivial. Instead, the existence of essentially monotonic bases will be characterised in terms of preferences conditional on non-null events belonging to a particular set of events. These non-null events play the role that states did in the previous section: the new isomorphic conditional preferences condition states that the preference orders conditional on these non-null events agree. However, for there to be an essentially monotonic basis, the preference orders conditional on *all* non-null events need to agree (Definition 2.4). To guarantee this, the non-null events used in the isomorphic conditional preferences condition need to have the appropriate stability property: the preference order conditional on any subevent must coincide with the preference order conditional on the event itself.

Definition 4.2. A non-null event E is *stable for* $c, d \in C$ if the following condition holds: $c \preccurlyeq_E d$ iff for every non-null event A such that $A \subseteq E$, $c \preccurlyeq_A d$. E is said to be *stable* if it is stable for all $c, d \in C$.

The following technical condition guarantees that an appropriate set of stable events exists.

Definition 4.3. *Local stability* holds if, for any $f, g \in \mathcal{A}$, there exists events E_1, E_2 and E_3, such that the non-empty E_i form a partition of S and such that: for any non-null event $A \subseteq E_1$, $f \prec_A g$; for any non-null event $A \subseteq E_2$, $f \sim_A g$; and for any non-null event $A \subseteq E_3$, $f \succ_A g$.

Proposition 4.1. *Suppose that local stability holds. Then there exists a partition $\mathcal{P}$ of S such that each non-null $E \in \mathcal{P}$ is stable.*

The partition in this proposition is exactly of the sort needed to characterise essential monotonicity. Where the condition proposed in the previous section concerns preference orders conditional on non-null states, the condition relevant here involves preference orders conditional on the non-null elements of the partition of stable events.

Definition 4.4. Assume that local stability holds and let $\mathcal{P}$ be a partition of S such that each $E \in \mathcal{P}$ is stable. Then *isomorphic conditional preferences* holds if for any non-null $E, E' \in \mathcal{P}$, $(C, \preccurlyeq_E)$ and $(C, \preccurlyeq_{E'})$ are isomorphic.[5]

It turns out that local stability is implied by essential monotonicity, so we have the following necessary and sufficient conditions for essential monotonicity in the Savage framework.

Theorem 4.1. *Essential monotonicity holds iff local stability and isomorphic conditional preferences hold.*

As in the Anscombe and Aumann case, this has a representation theorem as an immediate corollary.

Theorem 4.2. *Suppose that there is a representation of $\preccurlyeq$ of the sort specified in* (4.1) *which is unique up to similar positive affine transformations. Then*:

1. *The following are equivalent*:
   (i) *local stability and isomorphic conditional preferences hold and cardinal state independence holds with respect to an essentially monotonic basis $\mathcal{B}$*;
   (ii) *there exists a probability distribution p on S, a basis $\mathcal{B}$ and a real-valued function u on $\mathcal{B}$, such that, for $f, f' \in \mathcal{A}$, $f \preccurlyeq f'$ iff*

$$\int_S u\big(f^b(s)\big)\,dp \leqslant \int_S u\big(f'^b(s)\big)\,dp \tag{4.2}$$

   where f^b and f'^b are as in Proposition 2.1.
2. *For a given $\mathcal{B}$, p is unique, and u is unique up to positive affine transformations.*

A version of Corollary 3.1 continues to hold in the Savage framework, with mention of states s replaced by elements E of a partition of S consisting entirely of stable events. The remarks concerning Theorem 3.2 and Corollary 3.1 also apply here.

5. Discussion

As mentioned above, the idea of replacing the set of constant acts by a set of "really" constant acts has already been employed by Edi Karni, both in the Anscombe and Aumann framework

[5] Given that, for any pair of partitions with stable elements, each element of one will have a non-null overlap with at least one element of the other, isomorphic conditional preferences holds independently of the partition chosen, and so is well-defined.

[7] and in the Savage framework [8]. Leaving aside the technical differences between this paper and Karni's – Karni employs state-specific transformations of outcomes, whereas we use bases (Proposition 2.2 states that they are equivalent), Karni considers equivalence classes of outcomes whereas until now we have considered the outcomes themselves (but see below)[6] – there are fundamental differences between the goals of Karni's papers and this one.

In his work on the Anscombe and Aumann framework, Karni makes assumptions which are essential to his theorem and which imply that the set of outcomes is uncountably infinite[7]; by contrast, this paper only considers the finite and countably infinite cases. Moreover, his focus seems to be cardinal state independence, whereas the main concern here is ordinal state independence. Indeed, in the main result of his paper, he assumes monotonicity (Definition 2.3), and then constructs a basis which would be capable of supporting a state-independent representation; in the presence of monotonicity, the condition he places on this basis – state-invariance – corresponds to what we have called cardinal state independence (Definition 3.2).[8] These differences are perhaps not unrelated. When the set of outcomes is finite or countably infinite, the problem of finding a cardinally state independent basis is trivial in many cases, once the problem of finding an ordinally state independent basis has been solved (Corollary 3.1); ordinal state dependence thus seems a more natural subject of concern in the finite and countably infinite cases.

In the Savage case [8], Karni lays down some conditions on a basis; his main theorem states that, if there exists a basis satisfying these conditions, then there is a representation similar to (4.2). The main contrast between Karni's result and the representation theorem in Section 4 (Theorem 4.2) is the use of a condition requiring the existence of a particular basis in the former and the use of a condition requiring isomorphism of conditional preferences in the latter. From the viewpoint of standard economic methodology, there are two significant differences between these conditions.

Firstly, it is not immediately clear what property of the preference relation is demanded by the condition requiring the existence of a particular basis: it says that there is a set of acts such that the preferences over these acts have certain properties, but it does not specify which preference relations admit the existence of such sets of acts. By contrast, the condition on conditional preferences is formulated directly as a property of the preference relation: it says that the preference order given one stable event is the same as the preference order given any other stable event (in the sense of "the same" which is relevant for orders). This difference is significant if one supposes that the preference comparisons are observable from choices: it is not obvious what constraints Karni's condition implies on the agent's behaviour, whereas it can be straightforwardly checked, by comparing the agent's preferences conditional on different events, whether the condition proposed here is satisfied or not.

Secondly, the condition requiring the existence of a particular basis involves existential quantification over the objects of the theory: it quantifies across sets of acts, and the theory is dealing with preferences over a given set of acts. By contrast, the condition requiring isomorphic conditional preferences does not involve existential quantification over objects of the theory. At most, it involves existential quantification over technical constructions, insofar as the definition of isomorphism given above (Definition 3.3) involves quantification over mappings between orders. However, the notion of isomorphism can in fact be defined without using existential quantifica-

6 [7, p. 191] explicitly states this. [8] uses transformation functions that are onto though not necessarily one-to-one; these can be seen as bijective functions on equivalence classes.

7 He assumes the existence of a "conditional certainty equivalent" outcome for every consequence (p. 191).

8 He notes in the final section of his paper that the monotonicity assumption is not required for his theorem.

tion at all, if one employs the machinery of modern logic: two orders are isomorphic if everything you can say about the first (in an appropriately formalised language) holds for the second.[9] This difference has consequences for how the conditions are to be understood. Conditions which involve existential quantification over objects of the theory are usually thought of as "technical" or "structural": continuity axioms, which normally involve existential quantification over acts or events, are standard examples. Therefore, the condition requiring the existence of an appropriate basis is most naturally thought of as a technical condition, whereas the condition requiring isomorphic conditional preferences is not. As far as standard economic methodology is concerned, the result in Section 4 differs from that in [8] with respect to the possibility of applying the central conditions to agents' behaviour, and the interpretation of these conditions as technical or not.

Let us conclude by noting two extensions of the proposal made here. Firstly, since nothing in the isomorphic conditional preferences condition demands that the orders which are isomorphic are orders on the same set C, the results apply if it is assumed that there is a set C_s of outcomes for each state s.[10] If the orders $(C_s, \preccurlyeq_s)$ are isomorphic (and thus have the same cardinality), one can define a basis which "ties together" the different sets of outcomes and is essentially monotonic, so that, if cardinal state-independence holds as well, there is a representation of the form (3.3). A special case is when C_s is taken to be the set of equivalence classes of C under $\sim_s$. There may be cases where isomorphic conditional preferences does not hold for the initial set of outcomes, but does hold on the equivalence classes. The results above can be applied, yielding a representation of preferences over functions from states to equivalence classes of outcomes (or lotteries over equivalence classes of outcomes), which can be translated back into a representation of preferences over acts. This is the technique used in Karni [7,8].

Secondly, other results may be extended by weakening the monotonicity axiom in much the same way that Savage's and Anscombe and Aumann's theorems have been extended here. For example, several state-dependent utility theories have been proposed which assume monotonicity but drop the cardinal state independence axiom [10,14]. The characterisation theorems proved here provide an immediate generalisation of these results: monotonicity is replaced by isomorphic conditional preferences (and, in the Savage framework, local stability), and any weakened version of the cardinal state independence axiom is reformulated in terms of essentially monotonic bases.

Acknowledgments

The author would like to thank Philippe Mongin, Peter Wakker, two anonymous referees and the associate editor for useful comments, as well as the audience of FUR XIII (2008).

Appendix A

Proof of Proposition 2.1. Set $f^b(s) = b^i$ where b^i is the unique element of the $\mathcal{B}$ such that $b^i(s) = f(s)$. It remains to show that f^b is measurable. For each i and for each $c \in C$, $\{s \mid f(s) = c = b^i(s)\} = \{s \mid f(s) = c\} \cap \{s \mid b^i(s) = c\}$ and hence is measurable. For each i,

[9] An earlier version of this paper ("When is there state independence?", Cahiers de Recherche GREGHEC 883/2007, 2007) presented the result in this form. See any logic textbook, such as [2], for details.

[10] This is the sort of setup proposed in [5, §6], for example. In the text, the Anscombe and Aumann framework is discussed, but similar points hold for the Savage framework.

$\{s \mid f(s) = b^i(s)\} = \bigcup_{c \in C}\{s \mid f(s) = c = b^i(s)\}$ is a countable union of measurable sets and thus measurable. Hence f^b is measurable. □

Proof of Proposition 2.2. Let $\mathcal{B}$ be a basis. For $s, s' \in S$, define $\psi_{ss'} : C \to C$ as follows: for each $c \in C$, take the unique $b^i \in \mathcal{B}$ such that $b^i(s) = c$ and set $\psi_{ss'}(c) = b^i(s')$. It is readily checked that this is a well-defined bijective function and that the set $\Sigma = \{\psi_{ss'} \mid s, s' \in S\}$ has the required properties.

Let Σ be a family of the sort required. Pick any $t \in S$. Define the basis $\mathcal{B}$ as follows: the elements b^c are indexed by C, and $b^c(s) = \psi_{ts}(c)$. Since $\psi_{ss'}$ are bijections, $\mathcal{B}$ is a basis; moreover, it is readily checked that it has the required properties. □

Proof of Proposition 4.1. For each $c_j, c_k \in C$, there exists by local stability a partition into at most three elements which are stable for c_j, c_k. Call the partition $\mathcal{P}^{c_j c_k}$.

Let $\mathcal{Q}$ be the coarsest common refinement of $\mathcal{P}^{c_j c_k}$ for all $c_j, c_k \in C$.[11] Since C is at most countable, each element of $\mathcal{Q}$ is the intersection of at most countably many events; so it is an event. Furthermore, by construction, each non-null element of $\mathcal{Q}$ is a subevent of an event of $\mathcal{P}^{c_j c_k}$, for each $c_j, c_k \in C$. Since each element of $\mathcal{P}^{c_j c_k}$ is stable for c_j and c_k, each element of $\mathcal{Q}$ is also stable for c_j and c_k. This holds for all $c_j, c_k \in C$, hence every non-null element of $\mathcal{Q}$ is stable for all $c_j, c_k \in C$. □

The following lemma shall be useful in the proof of Theorem 4.1.

Lemma 1. *Suppose the Savage framework, so that S is infinite. If essential monotonicity holds, then local stability holds.*

Proof. Let $f, g \in \mathcal{A}$. By Proposition 2.1, there are corresponding measurable $f^b, g^b : S \to \mathcal{B}$; hence there are partitions $\mathcal{P}^f$, $\mathcal{P}^g$ of S such that f^b (respectively, g^b) takes constant values on each of the elements of $\mathcal{P}^f$ (respectively $\mathcal{P}^g$). Let $\mathcal{Q}$ be the coarsest common refinement of $\mathcal{P}^f$ and $\mathcal{P}^g$. On each element E of $\mathcal{Q}$, both f^b and g^b take constant values in $\mathcal{B}$; hence, since $\mathcal{B}$ is an essentially monotonic basis, $f \preccurlyeq_E g$ iff, for any non-null event $A \subseteq E$, $f \preccurlyeq_A g$. The partition required for local stability is readily constructed from $\mathcal{Q}$. □

The following lemma is at the heart of Theorems 3.1 and 4.1.

Lemma 2. *Suppose either the Anscombe and Aumann or the Savage framework, and suppose that there exists a partition $\mathcal{P}$ of S such that each non-null element of $\mathcal{P}$ is stable. Then essential monotonicity holds iff $(C, \preccurlyeq_E)$ and $(C, \preccurlyeq_{E'})$ are isomorphic for all non-null $E, E' \in \mathcal{P}$.*

Proof. *Left to right.* Suppose that essential monotonicity holds. Then there is a basis $\mathcal{B}$ such that, for any non-null event A and all $b^i, b^j \in \mathcal{B}$, $b^i \preccurlyeq b^j$ iff $b^i \preccurlyeq_A b^j$. So, for each non-null $E, E' \in \mathcal{P}$ and all $b^i, b^j \in \mathcal{B}$, $b^i \preccurlyeq_E b^j$ iff $b^i \preccurlyeq_{E'} b^j$. The identity map on $\mathcal{B}$ is an isomorphism between $(\mathcal{B}, \preccurlyeq_E)$ and $(\mathcal{B}, \preccurlyeq_{E'})$.[12] We show that, for any $E \in \mathcal{P}$, $(\mathcal{B}, \preccurlyeq_E)$ and $(C, \preccurlyeq_E)$ are isomorphic. This will be sufficient to prove the result, because it follows that, for any

[11] That is, the coarsest partition each of whose elements are contained in a single element of $\mathcal{P}^{c_j c_k}$ for each c_j, c_k.

[12] Just as for the case of $(C, \preccurlyeq)$, $(\mathcal{B}, \preccurlyeq')$ is the set $\mathcal{B}$ equipped with the order $\preccurlyeq'$.

$E, E' \in \mathcal{P}$, $(C, \preccurlyeq_E)$ is isomorphic to $(\mathcal{B}, \preccurlyeq_E)$, which is isomorphic to $(\mathcal{B}, \preccurlyeq_{E'})$, which is isomorphic to $(C, \preccurlyeq_{E'})$.

Fix $E \in \mathcal{P}$. Since each $b^i \in \mathcal{B}$ is measurable, the set $\{s \in E \mid b^i(s) = c\}$ is measurable, for all $c \in C$. So, to each b^i, there is an associated partition of E, $\mathcal{P}^{b^i} = \{\{s \in E \mid b^i(s) = c\} \mid c \in C\}$. Since C is at most countable, $\mathcal{B}$ is, and thus the coarsest common refinement of the $\mathcal{P}^{b^i}$ exists; call it $\mathcal{Q}$. By construction, for each non-null $A \in \mathcal{Q}$ and each $b^i \in \mathcal{B}$, $b^i(s) = b^i(s')$ for all $s, s' \in A$. So $\psi_A : \mathcal{B} \to C$, where $\psi_A(b^i) = b^i(s)$ for any $s \in A$, is a well-defined function. Since $\mathcal{B}$ is a basis, ψ_A is bijective; moreover, it trivially preserves the order $\preccurlyeq_A$. So, for each non-null $A \in \mathcal{Q}$, $(\mathcal{B}, \preccurlyeq_A)$ and $(C, \preccurlyeq_A)$ are isomorphic. Since $\mathcal{B}$ is a basis, for each non-null $A \in \mathcal{Q}$, $(\mathcal{B}, \preccurlyeq_A)$ is isomorphic to $(\mathcal{B}, \preccurlyeq_E)$; moreover, since E is stable, $(C, \preccurlyeq_A)$ is isomorphic to $(C, \preccurlyeq_E)$. Hence, $(\mathcal{B}, \preccurlyeq_E)$ is isomorphic to $(C, \preccurlyeq_E)$.

Right to left. Suppose that $(C, \preccurlyeq_E)$ to $(C, \preccurlyeq_{E'})$ are isomorphic for any $E, E' \in \mathcal{P}$. There thus exists a family Σ' of bijective order-preserving functions $\psi_{EE'} : C \to C$ for each $E, E' \in \mathcal{P}$ which is closed under composition and contains the identities. Construct a family Σ of bijective functions $\psi_{ss'} : C \to C$, for all $s, s' \in S$, as follows. If $s, s' \in E$ for some $E \in \mathcal{P}$, $\psi_{ss'}$ is the identity. If $s \in E$, $s' \in E'$ for some $E, E' \in \mathcal{P}$, $\psi_{ss'} = \psi_{EE'}$. It follows immediately that Σ is closed under composition and contains the identities. By Proposition 2.2, there is a unique basis $\mathcal{B}$ corresponding to Σ. It remains to show that $\mathcal{B}$ is an essentially monotonic basis.

For any non-null $E, E' \in \mathcal{P}$ and every $b^i, b^j \in \mathcal{B}$, $b^i \preccurlyeq_E b^j$ iff $b^i(s) \preccurlyeq_E b^j(s)$ for any $s \in E$ (since b^i, b^j are constant on E) iff $\psi_{EE'}(b^i(s)) \preccurlyeq_{E'} \psi_{EE'}(b^j(s))$ (since $\psi_{EE'}$ preserves order) iff $b^i(s') \preccurlyeq_{E'} b^j(s')$ for any $s' \in E'$ (since $\psi_{ss'} = \psi_{EE'}$ and $\psi_{ss'}(b^i(s)) = b^i(s')$) iff $b^i \preccurlyeq_{E'} b^j$. So, for each $b^i, b^j \in \mathcal{B}$, either $b^i \preccurlyeq_E b^j$ for all $E \in \mathcal{P}$ or $b^i \succ_E b^j$ for all $E \in \mathcal{P}$. By [12, Chapter 2, Theorem 2], it follows, in the former (respectively, latter) case, that $b^i \preccurlyeq b^j$ (respectively $b^i \succ b^j$)[13]; in other words, $b^i \preccurlyeq_E b^j$ for all $E \in \mathcal{P}$ iff $b^i \preccurlyeq b^j$. It remains to show that this is the case for any non-null event A. Since the events of $\mathcal{P}$ are stable, for every $E \in \mathcal{P}$, $b^i \preccurlyeq_E b^j$ iff $b^i \preccurlyeq_{E \cap A} b^j$ when $E \cap A$ is non-null. So, $b^i \preccurlyeq_E b^j$ for all $E \in \mathcal{P}$ iff $b^i \preccurlyeq_{E \cap A} b^j$ for all non-null $E \cap A$ with $E \in \mathcal{P}$. Applying [12, Chapter 2, Theorem 2] again, we have $b^i \preccurlyeq b^j$ iff $b^i \preccurlyeq_A b^j$; so $\mathcal{B}$ is an essentially monotonic basis. □

Remark *(Corollary 3.1).* The proof of the left-to-right direction has an immediate corollary that, for any two essentially monotonic bases $\mathcal{B}_1$ and $\mathcal{B}_2$ and for any non-null event E, $(\mathcal{B}_1, \preccurlyeq_E)$ and $(\mathcal{B}_2, \preccurlyeq_E)$ are isomorphic. If the order has a maximal or minimal element, it follows that, for each $b_1^i \in \mathcal{B}_1$, there exists $b_2^j \in \mathcal{B}_2$, such that, for any non-null event E, $b_1^i \sim_E b_2^j$. This implies Corollary 3.1.

Proof of Theorem 3.1. Immediate from Lemma 2, noting that, in the Anscombe and Aumann framework, the partition into singleton events has the required properties. □

Proof of Theorem 4.1. Suppose that essential monotonicity holds. By Lemma 1, local stability holds. Hence, by Proposition 4.1, there exists a partition of the sort required for Lemma 2 to hold; by the lemma, isomorphic conditional preferences holds. Suppose now that local stability and isomorphic conditional preferences hold. By Proposition 4.1, a partition of the sort required for Lemma 2 exists; by the lemma, essential monotonicity holds. □

[13] Although the theorem is stated by Savage, it is readily seen to apply in the Anscombe and Aumann framework.

References

[1] F.J. Anscombe, R.J. Aumann, A definition of subjective probability, Ann. of Math. Stat. 34 (1963) 199–205.
[2] C.C. Chang, H.J. Keisler, Model Theory, 3rd edn., Elsevier Science Publishing Company, Amsterdam, 1990.
[3] J.H. Drèze, Essays on Economic Decisions under Uncertainty, Cambridge University Press, Cambridge, 1987.
[4] P.C. Fishburn, Utility Theory for Decision Making, Wiley, New York, 1970.
[5] P.J. Hammond, Subjective expected utility, in: S. Barberà, P.J. Hammond, C. Seidl (Eds.), Handbook of Utility Theory, vol. 1, Kluwer, Dordrecht, 1998.
[6] B. Hill, An additively separable representation in the Savage framework, mimeo, GREGHEC, HEC Paris, 2008.
[7] E. Karni, A definition of subjective probabilities with state-dependent preferences, Econometrica 61 (1993) 187–198.
[8] E. Karni, Subjective expected utility theory with state dependent preferences, J. Econ. Theory 60 (1993) 428–438.
[9] E. Karni, P. Mongin, On the determination of subjective probability by choices, Manage. Sci. 46 (2000) 233–248.
[10] E. Karni, D. Schmeidler, On the uniqueness of subjective probabilities, Econ. Theory 3 (1993) 267–277.
[11] E. Karni, D. Schmeidler, K. Vind, On state dependent preferences and subjective probabilities, Econometrica 51 (1983) 1021–1032.
[12] L.J. Savage, The Foundations of Statistics, Dover, New York, 1954, 2nd edn., 1971.
[13] M.J. Schervish, T. Seidenfeld, J.B. Kadane, State-dependent utilities, J. Amer. Statist. Assoc. 85 (1990) 840–847.
[14] P. Wakker, H. Zank, State dependent expected utility for Savage's state space, Math. Oper. Res. 24 (1999) 8–34.