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Awareness Dynamics

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Abstract In recent years, much work has been dedicated by logicians, computer scientists and economists to understanding awareness, as its importance for human behaviour becomes evident. Although several logics of awareness have been proposed, little attention has been explicitly dedicated to change in awareness. However, one of the most crucial aspects of awareness is the changes it undergoes, which have countless important consequences for knowledge and action. The aim of this paper is to propose a formal model of awareness change, and to derive from it logics of awareness change. In the first part of the paper, the model of epistemic states of bounded agents proposed in Hill (Stud Log 89(1):81–109, 2008a) is extended with operations modelling awareness change. In the second part of the paper, it is shown how this model naturally extends the "standard" logic of awareness to yield a logic of awareness change.

Keywords Awareness · Knowledge · Logic of awareness · Awareness change · Belief revision · AGM belief revision · Dynamic epistemic logic

People lack awareness of various issues at particular moments. Their awareness (or lack of it) is an important property of their epistemic states, with significant consequences for their actions. Naturally, their states of awareness change, and rather often.

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Consider John, for example, who currently believes that a particular model of computer is a good deal. He has not considered whether it has fake parts; however, on learning that it does, he no longer believes it to be a good deal. He has not considered whether it comes with a particular word-processor. When the point is mentioned, he realises that it is relevant to his evaluation of whether the computer is a good deal or not (if it were not to come with the word-processor, he would not believe it to be a good deal), even though he has not been told whether it comes with the word processor or not. Finally, he has not considered whether or not the computer has a RAM S4T card; when this point is raised, it makes no difference to John's opinion (he being ignorant as to whether this card is a good or bad thing).

In all three cases, there seems to be a change in the issues which are in play for John; henceforth, we will say that there is a change in his *awareness*.[1] However, these changes seem to be of different sorts: in some cases, John revises his initial beliefs; in others cases, he recognises the relevance of the new issue to old questions; in others, he does not recognise any such relationship.

Although awareness has recently drawn the attention of computer scientists, logicians and game theorists (for example Halpern [9]; Modica and Rustichini [19]; Heifetz et al. [12]), a comprehensive understanding of awareness change is still lacking. For one, the little work which has been done to date on change of awareness has been in a game-theoretic or probabilistic setting [10, 13, 18], rather than a logical setting. Furthermore, there is no typology of the different sorts of awareness change, and no understanding of the relationship between awareness change and other changes of epistemic state, such as belief change. The aim of this paper is make a start at filling this gap, by developing both a model of, and logics for awareness change. Developing the model of an agent's epistemic state proposed in Hill [15], we will propose a formal set of operations which accurately represent the effect of change in awareness. Such a model will contribute to clarifying what is going on in changes of awareness – whether there is one or several sorts of awareness change, where the line between awareness change and belief change is, and so on. This is useful not only for decision theorists and game theorists, but also for philosophers and logicians looking to get a formal grip on real agents' epistemic states. On the basis of the model, logics of awareness change will be proposed in the Dynamic Epistemic Logic (DEL) paradigm [21]. In this paper, only the single agent case will be discussed.

[1] In the literature, awareness is sometimes reserved for cases where the issue of which the agent is unaware involves concepts he has not mastered [13]. The notion of unawareness employed here is broader, in that it encompasses all that the agent does not have in mind at the time (some might find the term 'attention' more appropriate). The use of this broader notion is justified, first of all, by the apparent similarity of the examples (only the case of RAM S4T would be a change in awareness in the stricter sense, since John already possesses the concepts of fake parts and word processor), and secondly, by the fact that a single model and set of operations can capture all these cases, as we shall see below.

In Section 1, the basic components of the model of an agent's epistemic state, encompassing awareness, which was developed in Hill [15] shall be recalled, and operations on this model representing change of awareness shall be proposed. In Section 2, a DEL-style logic of awareness and awareness change shall be proposed. A logic of awareness change in the AGM-paradigm [8] is given in Appendix A; proofs are presented in the Appendix B.

1 The Model

1.1 Preliminaries: A Model of Knowledge and Awareness

We, the theorists, know that he, the agent under study, is unaware of certain sentences.[2] The sentences available to the theorist are not all available to the agent; the languages at their disposal differ. This poses a problem for any theorist wishing the construct a *model* correctly representing the agent's epistemic state: is this model to be constructed in the agent's language or in the theorist's language? This question is equally important for the project of proposing a *logic* of knowledge and awareness: if, for example, the agent lacks the concepts of knowledge or awareness, the logic of his language will be significantly poorer than that of the theorist wishing to talk about these questions.

To represent the agent's epistemic state, we shall borrow and develop the model proposed and developed in Hill [15, 16]. This model is faithful to the agent's point of view, using his language, rather the theorist's language, in the representation of his beliefs. The crucial notions for our purposes are recalled below.

Definition 1 An *interpreted algebra* **B** is a triple (B_I, B, q), where B_I is the free Boolean algebra generated by a set I (the *interpreting algebra*),[3] B is an atomic Boolean algebra (the *base algebra*),[4] and $q : B_I \to B$ is a surjective Boolean homomorphism.

An *element* of **B** is a pair $(\phi, q(\phi))$, $\phi \in B_I$. Elements of an interpreted algebra shall be referred to (without risk of confusion) by the appropriate elements of the interpreting algebra, and shall often be called "sentences".

[2] Following recent practice, we shall suppose awareness to apply to sentences. See Board et al. [4] for a presentation of, and comparison with a theory of awareness of objects.

[3] A Boolean algebra is a distributed complemented lattice; the order will be written as $\leq$, meet, join, complementation and residuation as $\wedge, \vee, \neg, \rightarrow$. The free Boolean algebra generated by a set X shall be noted as B_X for the rest of the paper; details on this and other notions used in this paper may be found in Koppelberg [17].

[4] Standard terminology is employed here: an atom of a Boolean algebra is an element $a \in B$, such that, for all $x \in B$ with $\perp \leq x \leq a$, either $x = \perp$ or $x = a$.

The *consequence relation* $\Rightarrow_{\mathbf{B}}$ for an interpreted algebra $\mathbf{B}$ is defined as follows: for any elements ϕ, ψ of (B_I, B, q), $\phi \Rightarrow_{\mathbf{B}} \psi$ if and only if $q(\phi) \leq q(\psi)$. $\Leftrightarrow_{\mathbf{B}}$ will designate the derived equivalence relation. The subscripts may be dropped if they are evident from the context.

Finally, a *pointed algebra* is a pair $(\mathbf{B}, \chi)$ where $\mathbf{B} = (B_I, B, q)$ is an interpreted algebra and χ is an element of $\mathbf{B}$ such that $\chi \not\Leftrightarrow_{\mathbf{B}} \perp$. χ will be called the *knowledge element*.

Interpreted algebras represent the language and logical structure the agent is using at a particular moment. On the one hand, the interpreting algebra represents the language in play at that moment—the "local language"; the set I which generates it is the set of sentences taken by the agent to be primitive at that moment. On the other hand, the base algebra represents the logical structure on that language. As such, there is a natural relationship with "possible world structures": the atoms of the base algebra can be thought of as "small" possible worlds, in which only the sentences in the interpreting algebra are interpreted. Since we are dealing with finite agents, the algebras will be assumed to be finite.

Interpreted algebras naturally represent the agent's state of awareness at a given moment: awareness of a sentence ϕ is captured by the presence of ϕ in the interpreted algebra; unawareness of ϕ is captured by its absence.[5] Furthermore, they can represent what the agent is presupposing at a particular moment: he is presupposing a sentence ϕ if ϕ is equivalent to the $\top$ (under the consequence relation $\Rightarrow_{\mathbf{B}}$). In other words, the agent is presupposing ϕ if ϕ is true in all the small worlds in the interpreting algebra representing his state (at that moment). We shall illustrate these notions below; for further discussion, see Hill [15].

Pointed algebras model the state of explicit knowledge or belief[6] of the agent by an element χ of the interpreted algebra representing the language and logic he is using. The agent knows χ and all local consequences of χ (i.e. all ϕ such that $\chi \Rightarrow_{\mathbf{B}} \phi$). Putting aside the relativity to a local consequence relation $\Rightarrow_{\mathbf{B}}$, this representation of beliefs corresponds to the traditional representations as sets of possible worlds or consistent set of sentences closed under logical consequence; see Hill [16].

This model of the agent's epistemic state supposes that the agent's language is propositional: in particular, it does not contain knowledge or awareness operators. We call agents with propositional languages *simple agents*; by contrast, a *reflective agent* is one who has knowledge and awareness operators

[5]Since, as Dekel et al. [5] have shown, no non-trivial awareness operator can be defined on standard possible world structures, researchers have had to consider extensions of the standard framework to capture awareness [9, 12]; here, the relevant extension involves representing the agent's language explicitly.

[6]From an internal point of view, which is the point of view taken at this stage of the paper, the traditional notions of belief and knowledge in the logical literature coincide. They can thus be used interchangeably here. See also Section 2.1.

in his language, so that he can entertain sentences describing his own state of knowledge and awareness. The model can be easily extended to the case of reflective agents, by replacing Boolean algebras by "epistemic" algebras, which contain operators for knowledge and awareness.

Definition 2 An *epistemic algebra* B is a Boolean algebra with two operators: f_k, the knowledge operator, which satisfies normality, multiplicativity, T, 4 and 5,[7] and f_a, the awareness operator, with $f_a x = \top$, for all $x \in B$.

This yields the notion of interpreted epistemic algebra and pointed epistemic algebra.

Definition 3 An *interpreted epistemic algebra* $\mathbf{B}$ is a triple (B_I, B, q), where: B_I is the free epistemic algebra generated by a set I (the *interpreting algebra*), B is an atomic epistemic algebra (the *base algebra*), and $q : B_I \to B$ is a surjective epistemic homomorphism.

The knowledge and awareness operators in the interpreting algebra shall be respectively denoted by K and A. Elements and the consequence relation are as in Definition 1.

A *pointed epistemic algebra* is a pair $(\mathbf{B}, \chi)$ where $\mathbf{B}$ is an interpreted epistemic algebra and χ (the *knowledge element*) is an element of $\mathbf{B}$, not $\Rightarrow_{\mathbf{B}}$-equivalent to $\bot$, such that $\chi \Rightarrow_{\mathbf{B}} K\chi$ and if $\chi \not\Rightarrow_{\mathbf{B}} \psi$, then $\chi \Rightarrow_{\mathbf{B}} \neg K\psi$.[8]

K is the knowledge operator, satisfying the ordinary S5 axioms [6]; A is the awareness operator. With these operators in the language, the agent can consider—and explicitly know—sentences pertaining to his own state of knowledge and awareness. Since the agent can only consider those sentences of which he is aware, and he is aware of any sentence he considers, the agent's awareness operator in his local language at that moment is trivial: for any sentence in his local language at that moment, just being in that language implies that he is aware of it. Thus the condition on f_a in the definition of epistemic algebras. The comments following Definition 1 hold *mutatis mutandis* for pointed epistemic algebras.

As we will see in Section 1.2, very similar operations serve to model awareness changes in simple and in reflective agents, and, although the results in Section 2 will be stated only for reflective agents, they extend naturally to

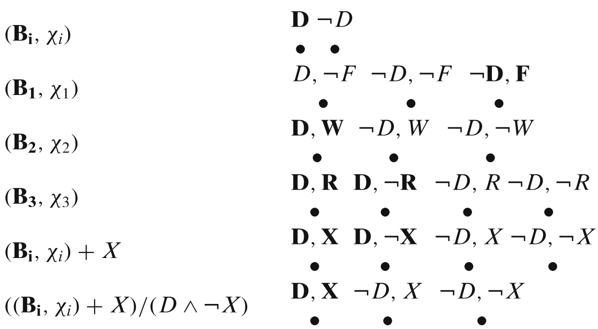
[7]The first two conditions correspond to the generalisation rule and the axiom K and are, respectively, $f_k\top = \top$ and $f_k(x \wedge y) = f_k x \wedge f_k y$. See Blackburn et al. [3, Ch 5]. The latter conditions are the algebraic formulations of the standard axioms: for example, T is $f_k x \leq x$. Analogous points to those made for Boolean algebras in footnote 3 apply to epistemic algebras.

[8]These conditions ensure that the knowledge set of the agent satisfies positive and negative introspection (axioms 4 and 5): the axioms common to both knowledge and belief. As noted in footnote 6, the traditional distinction between knowledge and belief cannot be drawn in a purely internal perspective: the difference is the truth axiom T, stating that what is known is really true, and one cannot talk about what is really true in a purely internal perspective. See also Section 2.1.

the case of simple agents. We take it to be an advantage of this framework and the approach taken in this paper that it can deal with richer and poorer agent languages, especially since both types of agents are relevant for different modelling situations (sometimes one considers one's state of knowledge, sometimes one is unaware of it).

Analysis of the example, part 1 Let us illustrate some of these notions on the example of John's computer purchase mentioned at the beginning of the paper. For this example, we may suppose that John is only using a propositional language (he is a simple agent). Pointed algebras modelling the initial states and the states after these changes are shown in Fig. 1. (In the figure, the small worlds of the base algebra are shown, labelled by the main sentences of the interpreting algebra which are true in them; the worlds where the knowledge element is true are in bold.) The simplest algebra which can model the initial state is $(\mathbf{B_i}, \chi_i)$. According to this algebra, he is aware of 'the computer is a good deal' (D) but not of 'the computer has fake parts' (F), 'the computer comes with a word-processor' (W), or 'the computer has a RAM S4T card' (R)—this is represented by the fact that the interpreting algebra contains the first sentence but none of the last three. Furthermore, neither D nor its negation is presupposed—there are worlds where D and worlds where $\neg D$. Finally, D is believed—the knowledge element implies D. The simplest algebra modelling the state after learning F is $(\mathbf{B_1}, \chi_1)$: he is aware of D and F (the interpreting algebra contains both); he accepts that $\neg D$ is a consequence of F (all F-worlds are $\neg D$-worlds); and he has learned F so no longer believes D (the knowledge element implies $F \wedge \neg D$). The simplest algebra modelling the state after learning W is $(\mathbf{B_2}, \chi_2)$: he is aware of D and W (the interpreting algebra contains both); he accepts that $\neg D$ is a consequence of $\neg W$ (all $\neg W$-worlds are $\neg D$-worlds); given that he has no information about $\neg W$, but retains his opinion about D, he must assume that W, for if not he would have to retract his belief in D (the knowledge element implies $W \wedge D$). The simplest algebra modelling the state after learning R is $(\mathbf{B_3}, \chi_3)$: there the interpreting algebra contains D and R; he accepts no relationship between D and R (all combinations of D and R are possible according to the logical consequence

Fig. 1 Some algebras involved in the analysis of the example



relation he is using); and he has no opinion about R but retains his opinion about D (the knowledge element implies D but neither R nor $\neg R$).

1.2 Changes of Awareness

1.2.1 Becoming Aware

The opening example of the paper contains three changes in John's state of knowledge which each involve him becoming aware of something and which have apparently different properties. Furthermore, they illustrate that changes of awareness can also be mixed in with changes in belief. The challenge is, firstly, to propose a principled way of distinguishing the aspects in these changes which have to do with knowledge and belief from those which have to do with awareness, and secondly, to provide a model for the apparent range of ways of becoming aware. We shall use the model described in the previous section to reply to these challenges.

First of all, consider the question of distinguishing components of attitude change which are changes in awareness from components which are changes of belief. According to the model presented above, awareness is characterised by a property of the local language in which his beliefs are couched, not of the logical structure on this language or of the beliefs themselves. So changes in which the agent just becomes aware of a new sentence correspond to changes in the epistemic state which are such that: (1) new sentences come into play but no sentences fall out of play; (2) there is no change in the logical relationships between the sentences of which he was already aware; (3) he does not change his state of knowledge or belief with respect to the old sentences, and he does not gain any beliefs not implied either by the logical relationships between the old sentences and the ones of which he becomes aware or by his own ignorance (for the case of a reflective agent).[9] Changes which involve both increases of awareness and changes of (explicit) belief satisfy (1) and (2) but violate (3). This gives the desired distinction: we shall say that a change in epistemic state is a *pure* case of becoming aware if satisfies (1)–(3) and that it is a *compound* case of becoming aware (by which we mean, a compound of a change in awareness and a change in belief) if it satisfies (1) and (2), but not (3).[10]

It turns out that changes in the agent's epistemic state satisfying (1)–(3) can be characterised formally as cases where the algebra modelling the initial state can be *embedded* in the algebra modelling the final state (Fact 1 in Appendix B), where embeddings are defined as follows.

[9]Note that, if the agent's set of beliefs did not change at all, then his set of beliefs would not be consistent and closed under logical consequence, by the lights of the new consequence relation of the enlarged language. As argued in Hill [16], we can assume that this does not occur. The case of the reflective agent will be discussed further below; see also Remark 3 in Appendix B.

[10]Changes which only violate (2) involve changes of awareness and changes in the presuppositions [15, 16] and fall beyond the scope of this paper.

Definition 4 An *embedding* of $\mathbf{B} = (B_I, B, q)$ into $\mathbf{B}' = (B_{I'}, B', q')$, with $I \subseteq I'$, consists of a pair of injective (epistemic) homomorphisms $\sigma_i : B_I \to B_{I'}$ and $\sigma_b : B \to B'$, the former generated by the inclusion of I in I', such that $q' \circ \sigma_i = \sigma_b \circ q$. This pair will be called σ. The embedding is said to be non-trivial if $\mathbf{B} \neq \mathbf{B}'$.

An embedding of $(\mathbf{B}, \chi)$ into $(\mathbf{B}', \chi')$ consists of an embedding of the interpreted (epistemic) algebras σ, such that $\chi' = \sigma(\chi)$ in the case of pointed algebras, and $\chi' = \sigma(\chi) \wedge \bigwedge_{\substack{\psi \in \mathbf{B}' \text{ non-epistemic} \\ s.t.\ \sigma(\chi) \not\Rightarrow_{\mathbf{B}'} \psi}} \neg K\psi$ in the case of pointed epistemic algebras.[11]

If there is a non-trivial embedding of a pointed (epistemic) algebra $(\mathbf{B_1}, \chi_1)$ into $(\mathbf{B_2}, \chi_2)$, we say that the latter is an *extension* of the former. In other words, a case of change is a pure case of becoming aware if and only if the pointed algebra representing the final state is an extension of the pointed algebra representing the initial state.

Analysis of the example, part 2 The first two parts of the example in the Introduction illustrate the distinction between pure and compound cases of becoming aware. As noted in Part 1 of the analysis, $(\mathbf{B_1}, \chi_1)$ and $(\mathbf{B_2}, \chi_2)$ in Fig. 1 represent the results of the changes involved in the example where John is told that the computer has fake parts and where he is made aware of the issue of the word processor respectively. In both cases, it is immediate that we have a case of becoming aware of a new sentence (F and W respectively), and that the properties (1) and (2) above are satisfied. However, only in the case of the word processor is property (3) satisfied: the word processor example is a pure case of becoming aware, whereas the example of the fake parts is a compound case. As noted in part 1 of the analysis of the example, in the word processor example, to retain his belief that the computer is a good deal (D), John is committed to acquiring the belief that it has a word processor (W). This behaviour does not contradict property (3), because the new belief follows from his initial beliefs, given the logical relationships between the old and new sentences: he accepts W because he accepts that W is implied by D and he believes D. In other words, this is not a change in belief which is driven by new information or considerations, but rather a consequence of the fact that John, whilst keeping the same basic beliefs, is using a different language with a new logical structure. Naturally, we are *not* suggesting that he should not revise his belief about the computer in this case—by for example withdrawing it until he acquires more information about the word processor—but only that this is properly analysed as involving a change in beliefs and not just a (pure) change in awareness. Indeed, something of this sort happens in the fake parts example.

[11] The non-epistemic elements of an epistemic algebra (B_I, B, q) are those which can be constructed from I only using Boolean connectives.

There, the (pure) increase in awareness yields a pointed algebra which looks just like ($\mathbf{B_2}$, χ_2) except with W replaced by $\neg F$; the subsequent change of knowledge element to yield ($\mathbf{B_1}$, χ_1), which comes from the fact that he learns that the computer has fake parts, is just a belief revision.

The example of the fake parts illustrates how the framework, with the notion of embedding, allows the factorisation of compound cases of becoming aware into pure awareness changes and belief changes. The possibility of such a factorisation seems to have been ignored in the literature; for example, Heifetz et al. [14] qualify the main example they discuss as an "awareness change" whereas, as for the case of the fake parts, it can be analysed as a pure awareness change followed by a straightforward belief change. However, the possibility of factorisation is important, for it allows us to directly apply the large body of research on belief change, in logic, philosophy and economics, to such examples.

In the rest of the paper, we focus entirely on pure cases of awareness change. Concerning these cases, the notion of extension is not entirely satisfactory as an operation of becoming aware, because it is relational (multi-valued) rather than functional (single-valued). This corresponds to the fact that there are several ways of (purely) becoming aware of a new sentence. The example of word processor indicates one: the new sentence may enter into non-trivial relationships with sentences of which the agent was previously aware – if the computer did not come with a word processor, it would no longer be considered a good deal. The example of the RAM S4T card illustrates another: here, there is no relationship between the new sentence and the ones of which he was initially aware—whether or not the computer has a RAM S4T card, he will still consider it a good deal.

The framework presented in the previous section allows one to further analyse the range of pure cases of becoming aware by performing another factorisation. Let us say that a pure increase in awareness is *free* if there are no logical relationships (for the agent) between the new sentences and the old ones. The idea is to factorise the range of pure cases of becoming aware into a single basic operation for freely becoming aware of a new sentence, and an operation for establishing logical relationships between sentences. This is an economic way of coping with the range of ways of becoming aware: there are just two operations—one determinate operation for free cases of becoming aware, and an operation for establishing logical relationships between the new sentences and the old ones which can be applied in a variety of ways.

The operation on interpreted algebras which models free cases of becoming aware, called *expansion*, simply involves an extension of the language of the interpreted algebra with no non-trivial logical structure added.

Definition 5 The *expansion* of an interpreted (epistemic) algebra $\mathbf{B} = (B_I, B, q)$ by a sentence ϕ, call it $\mathbf{B} + \phi$, is defined as follows: if $\phi \notin B_I$,

$\mathbf{B} + \phi = (B_{I \cup \{\phi\}}, B', q')$, where $B' = B \otimes B_{\{x\}}$[12] and q' is the (epistemic) homomorphism such that $q'(\phi) = x$ and $q'(\psi) = q(\psi)$ for all $\psi \in B_I$. If $\phi \in B_I$, $\mathbf{B} + \phi = \mathbf{B}$.

The expansion of the pointed algebra $(\mathbf{B}, \chi)$ by ϕ, written as $(\mathbf{B}, \chi) + \phi$, is $(\mathbf{B} + \phi, \chi)$.[13] The expansion of the pointed epistemic algebra $(\mathbf{B}, \chi)$ by ϕ, written as $(\mathbf{B}, \chi) + \phi$, is $(\mathbf{B} + \phi, \chi \wedge \bigwedge_{\substack{\psi \in \mathbf{B}\ non-epistemic \\ s.t.\ \chi \nRightarrow_{\mathbf{B}} \psi}} \neg K(\phi \vee \psi) \wedge \neg K(\neg \phi \vee \psi))$.

As one would expect from the preceding discussion, expansion does not alter the knowledge element in the case of pointed algebras. The case of pointed epistemic algebras is complicated by the well-known issues arising from the presence of the operators for knowledge in the language [7, 22]. Since the agent must know his attitude—of knowledge or ignorance—toward any new sentence of which he becomes aware, he must in a certain sense "gain" knowledge of his own knowledge or ignorance of a new sentence on becoming aware of it. Since expansion is intended to be the basic operation of becoming aware and nothing else, the agent is assumed to be ignorant of the truth of the new non-epistemic sentences of which he becomes aware (he neither believes them to be true nor believes them to be false); his knowledge only augments by his knowledge of this ignorance.

The operation of *restriction* will be used to model the non-free cases, where the newly added sentence enters into logical relationships with existing sentences (insofar as they feature in the local logical structure effective at that moment).

Definition 6 The *restriction* of an interpreted algebra $\mathbf{B} = (B_I, B, q)$ by ϕ, written $\mathbf{B}/\phi$, is (B_I, B', q'), where $B' = B/q(\phi)$[14] and q' is the composition of this quotient with q. The restriction of the pointed algebra $(\mathbf{B}, \chi)$ by ϕ, written as $(\mathbf{B}, \chi)/\phi$, is $(\mathbf{B}/\phi, \chi)$. The restriction of an interpreted epistemic algebra $\mathbf{B} = (B_I, B, q)$ is $\mathbf{B}/\phi = (B_I, B', q')$, where B' is the modal algebra corresponding to the relational structure obtained by restricting the relational structure corresponding to the algebra B to the $\neg q(\phi)$ states and removing

[12] Recall (footnotes 3 and 7) that $B_{\{x\}}$ is the free Boolean (respectively epistemic) algebra generated by the single element x. The symbol $\otimes$ denotes the free product operation. The free product of (atomic) Boolean algebras is the Boolean algebra whose set of atoms is the Cartesian product of the sets of atoms of the original Boolean algebras; the free product of (atomic) epistemic algebras is the epistemic algebra whose set of atoms is a set of copies of subsets of the Cartesian product of the sets of atoms of the original algebras, where each subset contains at least one element corresponding to each of the atoms of the original algebras and the copies exhaust the set of relations on such subsets such that any two atoms are connected if and only if their components are connected as atoms of the original algebras. Several of the operators used in this section are discussed in Hill [15].

[13] Here and throughout the paper, the same symbol (in this case χ) shall be used both for the element of original algebra ($\mathbf{B}$) and its image (in this case, its image under the natural embedding of $\mathbf{B}$ into $\mathbf{B} + \phi$), since the difference is clear from the context.

[14] $B/q(\phi)$ is the quotient of the Boolean algebra B by the element $q(\phi)$, that is, the quotient by the smallest congruence relation $\sim$ on B such that $q(\phi) \sim \bot$ (see Hill [15, Example 2]).

bisimilar copies of cells, and q' is the modal homomorphism which agrees with q regarding the valuation of the elements of I on the common states. The restriction of the pointed epistemic algebra $(\mathbf{B}, \chi)$ by ϕ, written as $(\mathbf{B}, \chi)/\phi$, is $(\mathbf{B}/\phi, \chi')$ for $\chi' = K\bar{\chi} \wedge \bigwedge_{\substack{\psi \text{ non-epistemic s.t.} \\ K\bar{\chi} \not\Rightarrow_{\mathbf{B}/\phi} \psi}} \neg K\psi$, where $\bar{\chi}$ is the $\Rightarrow_{\mathbf{B}}$-minimal non-epistemic element such that $\chi \Rightarrow_{\mathbf{B}} \bar{\chi}$.[15]

In the algebra after restriction, there is new logical structure. This operation does not alter the language the agent is using (the interpreting algebra) but only the structure on that language; hence it does not alter awareness. Since the logical consequence relation becomes stronger, more might follow from his current knowledge, so the set of sentences which he knows or believes increases. However, as with the case of the word processor discussed above, this is not a change in the set of beliefs that violates property (3), since all new beliefs follow from the initial beliefs under the new consequence relation: it is the consequence relation more than the beliefs themselves which has changed. The structural change in the algebra corresponds to the removal of all small worlds where ϕ; indeed, the restriction operation on interpreted epistemic algebras is basically the operation involved in models of public announcement [21]. Since these models have been studied elsewhere, little specific attention shall be dedicated to this operation in what follows.

The claim is that any pure case of becoming aware (extension) can be factorised into free cases of becoming aware (expansions) and changes in the logical structure (restrictions). This is expressed by the following proposition.

Proposition 1 *Let σ be an embedding of $(\mathbf{B_1}, \chi_1)$ into $(\mathbf{B_2}, \chi_2)$ (both either pointed or pointed epistemic algebras). Then there are $\phi_1, \ldots, \phi_n$, and $\psi \in (\mathbf{B_1} + \phi_1 + \cdots + \phi_n) \setminus \mathbf{B_1}$ such that $(\mathbf{B_1}, \chi_1) + \phi_1 + \cdots + \phi_n/\psi = (\mathbf{B_2}, \chi_2)$.*

We will say that a pointed (epistemic) algebra $(\mathbf{B_2}, \chi_2)$ is an *extension of* $(\mathbf{B_1}, \chi_1)$ *by* $\phi_1, \ldots, \phi_n$ $(n \geq 1)$ if and only if there is a sentence $\psi \in (\mathbf{B_1} + \phi_1 + \cdots + \phi_n) \setminus \mathbf{B_1}$ such that $(\mathbf{B_2}, \chi_2) = (\mathbf{B_1}, \chi_1) + \phi_1 + \cdots + \phi_n/\psi$.

Analysis of the example, part 3 To illustrate these notions, first consider the change involved in the example of the RAM S4T: here a new sentence is brought into play, but it does not enter into any relationship with the old sentences. This can be modelled by an expansion of the algebra interpreting the initial state $((\mathbf{B_i}, \chi_i))$ by the sentence R. In Fig. 1, the expansion of $(\mathbf{B_i}, \chi_i)$ by a sentence X is shown; substituting R for X, this is visibly equivalent to $(\mathbf{B_3}, \chi_3)$.

[15] This definition tacitly uses results showing the relationships between relational structures and modal algebras [3, Ch 5]. For the finite case, which is the main one considered here, not only does each relational structure generate a modal algebra, but each model algebra generates a corresponding relational structure. For more on the relational notions used, such as bisimulation, see Blackburn et al. [3] and van Ditmarsch et al. [21]. A purely algebraic version of the definition would require us to enter into considerations orthogonal to the topic of this paper.

The change in the word processor example involves the bringing into play of a new sentence which enters into logical relationships with ones of which John was already aware. This can be modelled by an expansion by W, followed by a restriction by $D \wedge \neg W$ (the computer cannot both not come with a word processor and be a good deal). The result of an expansion by X followed by a restriction by $D \wedge \neg X$ is given in Fig. 1; substituting W for X, it is visibly equivalent to $(\mathbf{B_2}, \chi_2)$. As noted in part 2 of the analysis, the change involved in the case of fake parts is just this awareness change, followed by belief change.

The notions and factorisations introduced in this section are summarised in Fig. 2 (where the arrows mean: 'may be factorised into').

1.2.2 Becoming Unaware

Not only can one become aware of a sentence, but a sentence can also fall out of one's awareness. The framework proposed above also has a simple story to tell about these sorts of changes: since awareness is the presence of a sentence in the interpreted algebra, becoming unaware involves the loss of the sentence, and the structure associated with it, from the algebra. However, just as in the case of becoming aware, this loss does not affect the beliefs regarding sentences which the agent stays aware of or the logical structure on these sentences; cases where there are such changes are not pure cases of becoming unaware, but rather compound changes, consisting of a change of awareness combined with a belief revision, say. Rather than repeat the distinctions between compound and pure change here, we shall directly introduce the operation which models pure cases of becoming unaware.

Definition 7 The *narrowing* of an interpreted (epistemic) algebra (B_I, B, q) by ϕ, written $\mathbf{B} - \phi$, is defined as follows: if $\phi \in B_I$, $\mathbf{B} - \phi = (B_{I \setminus f(\phi)}, B', q')$, where $f(\phi)$ is the smallest $I' \subseteq I$ such that $\phi \in B_{I'}$, q' is the restriction of q to $B_{I \setminus f(\phi)}$ and B' is the image of $B_{I \setminus \Phi}$ under q. If $\phi \notin \mathbf{B}$, $\mathbf{B} - \phi = \mathbf{B}$.

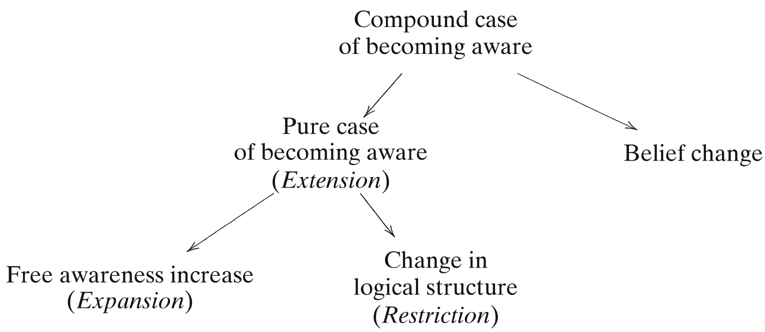


Fig. 2 Notions of becoming aware

The narrowing of the pointed (epistemic) algebra $(\mathbf{B}, \chi)$ by ϕ, written as $(\mathbf{B}, \chi) - \phi$, is $(\mathbf{B} - \phi, \psi)$, where ψ is the $\Rightarrow_{\mathbf{B}-\phi}$-minimal element such that $\chi \Rightarrow_{\mathbf{B}} \psi$.[16]

The case of becoming unaware differs from the case of becoming aware—and indeed from many traditional sorts of changes of epistemic states such as belief change—in that it is more difficult to pin down a trigger for the change and a moment when it occurs. Becoming unaware is a gradual, ephemeral process, and cannot be thought to be explicitly and directly triggered by some announcement or event: one cannot be told to become unaware of something, as one can be told to become aware of it, to believe it or to cease to believe it. The operation of narrowing can only be thought of as a formal representation of the process by which the state of awareness changes. This is all that is required for it to be able to be used in analyses of changes in epistemic state, such as the sorts of analyses undertaken in the previous section using becoming-aware operations. Indeed, many of the main points made in the previous section apply: in particular, a change in which one becomes unaware of a sentence often does not occur in isolation, but combined with a change in belief, say. As for the case of the becoming-aware operations proposed above, the framework yields a natural decomposition of such changes into an awareness-change component—in this case, the narrowing operation—and a belief-change component, which can be formalised using the appropriate belief change operation.

One consequence of the lack of explicit trigger for cases of becoming unaware which does seem to differentiate it from cases of becoming aware and belief change is that narrowing need only be defined for primitive sentences (elements in I), rather than for any sentence in the interpreted algebra. Given the closure assumptions on the set of sentences of which the agent is aware (see Section 2.1 and Hill [15]), any case of becoming unaware boils down to the removal of a certain number of sentences from the generating set I of the agent's language, with the consequence that sentences formed using these primitive sentences are also removed. Since the trigger for the change is not assumed to be given but rather provided by the theorist in the characterisation he offers of the change, he can always decide to model such a change as a case of narrowing by these elements of I. Therefore, the narrowing operation need only be defined on elements of I. The above definition, applied to a single element of I, behaves as one would expect: the resulting algebra is the subalgebra generated by the other elements of I. The definition is only extended to other elements of the algebra for the sake of completeness; this is done by supposing that they characterise a change in which the agent becomes unaware of all the primitive sentences they (essentially) contain.

[16]Note that this definition is correct, because we are supposing finite algebras (Section 1.1). This supposition is stronger than required: algebras containing meets for any set of elements would be sufficient.

2 The Logic

In this section, we characterise the awareness change operations proposed above. In other words, we give a logic of awareness change using the model in Section 1. Opinions differ as to what counts as a logic of a change operation. Considering, for the sake of comparison, the case of belief revision, there are (at least) two paradigms: the AGM paradigm initiated by Alchourron et al. [1], and the DEL paradigm, building on work on dynamic extensions of logics of knowledge [21]. The former paradigm traditionally does not assume that there are belief operators in the object language, leaving talk of belief and belief change only in the metalanguage [8]. As such, the object language is most naturally thought of as the agent's language rather than the theorist's language. The latter paradigm, by contrast, adopts a richer object language, containing not only operators for belief but also operators describing the changes in belief which may occur [2, 20, 22]. Whereas in the case of belief revision, it may be possible to think of this object language as the agent's language, this is impossible, on pain of triviality, in the case of awareness and awareness change. After all, as noted in Section 1.1, the agent's awareness operator is trivial: he cannot talk about sentences of which he is unaware.[17] Adopting the DEL paradigm is only interesting if one studies the theorist's language.

In this section, we will adopt the latter paradigm and propose a dynamic logic for awareness change. (An AGM-style logic for awareness change is presented in Appendix A.) Dynamic logics function by building on an underlying static logic [21]: for example, Dynamic Epistemic Logic builds on standard S5 epistemic logic. Our logic of awareness change will be built on the best candidate for a "standard" logic of awareness. We will first remind the reader of this logic, and show that the model presented above provides a (new) semantics for it. Then we will present a dynamic extension—the logic of awareness change—and show it to be sound and complete with respect to the change operations proposed in Section 1.2.

2.1 Preliminaries: The Logic of Awareness

Let $\mathcal{L}_P^{KA}$ be the language generated from a set of propositional letters P by closure under Boolean connectives and the operators K (explicit knowledge) and A (awareness). The system $\mathbf{Aw^r} = \{Prop, MP, Gen, K, T, 4, 5, A0 - A5\}$ (see Fig. 3 for axioms and rules) can currently be thought of as the "standard" logic of awareness: it is sound and complete over several important semantics for awareness that have been recently proposed, notably Halpern's propositionally determined awareness structures [9, 11] and the generalised standard models of Modica and Rustichini [19].

[17] Evidently, consideration of operators in the agent's language describing the awareness of others is of interest; as noted at the outset, this is beyond the scope of the current paper.

Static axioms and rules

Prop	Axioms of propositional logic	**A0**	$K\phi \to A\phi$
MP	Modus ponens	**A1**	$A(\psi \wedge \phi) \leftrightarrow A\phi \wedge A\psi$
Gen	From ϕ infer $A\phi \to K\phi$	**A2**	$A\phi \leftrightarrow A\neg\phi$
K	$K\phi \wedge K(\phi \to \psi) \to K\psi$	**A3**	$AK\phi \leftrightarrow A\phi$
T	$K\phi \to \phi$	**A4**	$A\phi \leftrightarrow AA\phi$
4	$K\phi \to KK\phi$	**A5**	$A\phi \to KA\phi$
5	$\neg K\phi \wedge A\neg K\phi \to K\neg K\phi$		

Satisfaction clauses

$((\mathbf{B}, \chi), \Xi) \vDash \phi$	iff	$\phi \in \Xi$
$((\mathbf{B}, \chi), \Xi) \vDash \neg\phi$	iff	$((\mathbf{B}, \chi), \Xi) \nvDash \phi$
$((\mathbf{B}, \chi), \Xi) \vDash \phi \wedge \psi$	iff	$((\mathbf{B}, \chi), \Xi) \vDash \phi$ *and* $((\mathbf{B}, \chi), \Xi) \vDash \psi$
$((\mathbf{B}, \chi), \Xi) \vDash K\phi$	iff	$\chi \Rightarrow_{\mathbf{B}} \phi$ *and* $\neg\phi \notin \Xi$
$((\mathbf{B}, \chi), \Xi) \vDash A\phi$	iff	$\phi \in \mathbf{B}$

Fig. 3 Logic of awareness

With a view to using this static logic to develop a dynamic logic based on the change operations proposed in Section 1.2, let us note that the model presented in Section 1.1 can be used to provide a sound and complete semantics for $\mathbf{Aw^r}$. The basic idea is simple: since $\mathcal{L}_P^{KA}$ is the theorist's language for talking about the agent's knowledge and awareness, and the pointed (epistemic) algebras presented above model the agent's epistemic state, the sentences of the theorist's language which refer to the agent's state of knowledge and awareness will be *interpreted* on the pointed (epistemic) algebras modelling the agent's epistemic state. For example, ϕ is in the algebra modelling the agent's epistemic state if and only if the agent is aware of ϕ; so the theorist's sentence $A\phi$ is true if and only if ϕ is in the appropriate algebra.

This idea has two technical consequences. Firstly, it gives a way of interpreting sentences about the agent's knowledge and awareness, but it does not give an interpretation of sentences in the theorist's language which do not talk about the agent's knowledge and awareness. Indeed, sentences about what is really the case cannot and should not be interpreted in models of the agent's epistemic state. If these sentences are part of the theorist's language, extra structure, representing the actual state of the world, will be required to interpret them. The formulae not containing epistemic operators will be interpreted in the classical way, on maximally consistent sets of propositions of the propositional language $\mathcal{L}_P$ generated by P; these are to be considered as the sets of sentences which are actually true.

The second point concerns the models of the agent's epistemic states which should be used. The standard languages for the logic of awareness, $\mathcal{L}_P^{KA}$, contain sentences with embedded operators: sentences expressing the agent's knowledge about his knowledge, his knowledge about his awareness, and so on. Interpretation of such sentences is straightforward if the agent is reflective (i.e. has knowledge and awareness operators in his language): $KK\phi$ is true if the agent believes $K\phi$ and $K\phi$ is indeed true (where $K\phi$ is a sentence of the agent's language), and so on. However, how are such sentences to be interpreted if the agent is simple (i.e. has no knowledge or awareness operator in his language)? This is a subtle philosophical point, but there are reasons to think that such sentences cannot be interpreted, and perhaps do not make sense. After all, although one can, via theories of behaviour, give an account of belief, and perhaps knowledge, such that an observer can attribute belief or knowledge to an agent who does not possess the concept of belief or knowledge (for example, the agent believes ϕ if he acts in accordance with a belief that ϕ), it is much less clear how one could give an account such that the observer can attribute second-order belief or knowledge to an agent devoid of the concept. What is it to act in accordance with the belief that he believes ϕ, if he does not even possess the concept of belief? Given this, the language $\mathcal{L}_P^{KA}$ is the language the theorist should use only if he is talking about a reflective agent; we will thus use pointed epistemic algebras to interpret this language. (Nevertheless, the framework and results below extend directly to simple agents, by using pointed algebras in the semantics, restricting the language to formulae of modal depth at most one, and removing axioms involving formulae of modal depth greater than one.)

So $\mathcal{L}_P^{KA}$ will be interpreted on pairs $((\mathbf{B}, \chi), \Xi)$, where $(\mathbf{B}, \chi)$ is a pointed epistemic algebra whose interpreting epistemic algebra is generated by a subset of P and Ξ is a maximally consistent set of sentences of the propositional language $\mathcal{L}_P$ generated by P. Let $\mathcal{B}^e$ be the class of all such pairs. The interpretation is defined by the satisfaction clauses given in Fig. 3. They are generally as anticipated in the preceding discussion, and require only one supplementary remark. Recall that the knowledge operator in the agent's language is ambiguous between belief and knowledge (Section 1.1, in particular footnote 6). However, the knowledge operator in the theorist's language does not exhibit this ambiguity: knowledge is distinguished from belief (according to the standard sense used by modal logicians) in that the former implies truth in the world, whilst the latter does not. The second conjunct in the clause for K ensures that K is interpreted as knowledge. Removing this conjunct, one would obtain a logic for awareness and belief, rather than awareness and knowledge.[18]

This is a sound and complete semantics for the logic of awareness.

Theorem 1 *The system* $\mathbf{Aw^r} = \{Prop, MP, Gen, K, T, 4, 5, A0 - A5\}$ *is sound and complete with respect to* $\mathcal{B}^e$.

Remark 1 A noteworthy property of Theorem 1 is the dependence on the stipulation that the interpreting algebras of the pointed epistemic algebras in $\mathcal{B}^e$ are generated by subsets of P, the set of primitive sentences of $\mathcal{L}_P^{KA}$. In a word, this condition demands that the agent's language always agrees with the

[18]Given the properties of epistemic algebras (Definition 3), reference to truth in the world is only required for non-epistemic sentences; the use of the double negation in the clause ensures that it only places restrictions on these sentences.

theorist's concerning the sentences taken as primitive; it follows that the consequence relation he is using can always be taken to agree with the theorist's. If this condition is dropped, the agent's local consequence relation, under which his knowledge set is closed, may not agree with the theorist's. Given that the axioms are expressed in the theorist's language with the theorist's consequence relation, the agent's state of knowledge may cease to satisfy some of these axioms. The rule of generalisation and the axiom K, for example, may suffer. If the agent takes the sentences ϕ and $\phi \rightarrow \psi$ to be primitive, then it need not be the case that $\phi \wedge (\phi \rightarrow \psi) \Rightarrow \psi$ in his interpreted algebra (even if ψ belongs to this algebra). Therefore, he could have a knowledge element according to which he knows ϕ and $\phi \rightarrow \psi$ but not ψ, or, to put it another way, according to which he does not know $(\phi \wedge (\phi \rightarrow \psi)) \rightarrow \psi$, although he is aware of it; so K and generalisation fail. See Hill [15, §1] for a fuller discussion of this issue.

This point highlights the difference between the task of (formally) modelling a phenomenon such as knowledge or awareness and the task of giving a logic of the phenomenon. The problem of primitive sentences concerns the relationship between the theorist's and the agent's languages, and thus only arises when one tries to develop a logic of (the agent's) awareness, which, for fear of triviality, needs to be expressed in the theorist's language. It indicates that when one's aim is to provide a model of the agent's state of knowledge and awareness which is as accurate as possible, use of the theorist's language should perhaps be avoided, and thus logics of the sort discussed above may be inappropriate. On the other hand, the use of the theorist's language has the advantage of rendering logics more tractable. These considerations suggest that one should admit a sharp distinction between models of awareness and logics of awareness, and they argue for the development of models which rely as little as possible on the theorist's language and for an account of the relationship between our best models for awareness and logics of awareness. In other words, they support the strategy of the current paper, the model proposed in Section 1, and the project undertaken in this section of developing logics based on this model.

2.2 The Logic of Awareness Change

The appropriate theorist's language for talking about awareness change will be an extension of the language $\mathcal{L}_P^{KA}$ containing operators $[+\phi]$ and $[-\phi]$. We will give two axiom systems for awareness change: in one, $[+\phi]$ will be understood as 'after a pure case of becoming aware of ϕ', and in the other, it will be understood as 'after freely becoming aware of ϕ'. In both, $[-\phi]$ will be understood as 'after a pure case of becoming unaware of ϕ'.

As in the discussion of embedded formulae in Section 2.1, the agent's expressive capacities place restrictions on the things which the theorist can meaningfully say about his changes of awareness. Most notably, the agent's language does not contain operators for talking about change of awareness. Although this restriction may be unnecessary for the case of becoming unaware—after all, one can consider what one would think had one not been aware of certain issues—it is certainly natural for the case of becoming aware—to consider what one would think had one been aware of a particular issue, one must already be aware of it. Given this assumption on the agent's language, it makes no sense for the theorist to consider sentences such as 'after becoming aware of "after becoming aware of ϕ, ψ", χ' or 'the agent knows that, after becoming aware of ϕ, ψ'. For this reason, the language used here will not contain (1) sentences where there is an operator $[+\phi]$ or $[-\phi]$ with ϕ having a subformula $[+\psi]\psi'$ or $[-\psi]\psi'$; and (2) sentences where there is an operator $[+\phi]$ or $[-\phi]$ in the scope of K or A. Let $\mathcal{L}_P^{KA+-}$ be the extension of the language $\mathcal{L}_P^{KA}$ by operators $[+\phi]$, $[-\phi]$ for all $\phi \in \mathcal{L}_P^{KA}$ which contains only sentences where these operators are not in the scope of K and A.[19]

For the remainder of this section, $f(\phi) = \{p_1, \ldots, p_n\}$ is taken to denote the set of elements of P featuring in ϕ, for $\phi \in \mathcal{L}_P^{KA}$. To interpret the new operators, the appropriate clauses in Fig. 4 are added to those presented in the previous section. $[-\phi]$ is understood to refer to a pure case of becoming unaware, and is thus interpreted by the narrowing operation. If $[+\phi]$ is understood as a free case of becoming aware, the interpretation is as given under "Expansion" and involves the expansion operation. (The definitions of expansion and narrowing are extended to elements of $\mathcal{B}^e$ by stating that they do not alter the set of sentences Ξ.) If $[+\phi]$ is interpreted as a pure case of becoming aware, the interpretation is as given under "Extension" and involves extensions. Recall that the notion of extension is relational, rather than functional; the satisfaction clause is thus most naturally defined with respect to a function which picks out one extension for each algebra and sentence. Let $\mathcal{B}^e_{ext}$ be the set of pairs of elements of $\mathcal{B}^e$ and functions γ_{ext} which associate to every $((\mathbf{B}, \chi), \Xi) \in \mathcal{B}^e$ and every sentence ϕ an element $((\mathbf{B}', \chi'), \Xi) \in \mathcal{B}^e$ where $(\mathbf{B}', \chi')$ is an extension of $(\mathbf{B}, \chi)$ by the primitive sentences $p_1, \ldots, p_n \in f(\phi)$. The satisfaction clause for extension is defined on elements of $\mathcal{B}^e_{ext}$ as shown in Fig. 4. (The other satisfaction clauses in Figs. 3 and 4 are extended to $\mathcal{B}^e_{ext}$ in the natural way.)

The axioms for dynamic operators are also given in Fig. 4. Let $\mathbf{Aw}^{\mathbf{rd_{ext}}}$ be the result of extending $\mathbf{Aw}^{\mathbf{r}}$ by {+ Gen, + prop, + ∧, + ¬, + A, + K, – prop, – ∧, – ¬, – A, – K}, and let $\mathbf{Aw}^{\mathbf{rd_{exp}}}$ be the result of extending $\mathbf{Aw}^{\mathbf{rd_{ext}}}$ by $+_{exp}$ K. We have the following pair of soundness and completeness results.

Theorem 2 *The system* $\mathbf{Aw}^{\mathbf{rd_{ext}}}$ *is sound and complete with respect to* $\mathcal{B}^e_{ext}$. *Moreover, the system* $\mathbf{Aw}^{\mathbf{rd_{exp}}}$ *is sound and complete with respect to* $\mathcal{B}^e$.

[19] In fact, the semantics and axiomatisation can be extended to languages which admit embeddings of the operator $[-\psi]$, without having to alter the models so that the agent's language includes such operators. It suffices to use the evident reductions of $A[-\psi]\phi$ and $K[-\psi]\phi$, namely: $A[-\psi]\phi \leftrightarrow A\psi \wedge A\phi$ and $K[-\psi]\phi \leftrightarrow [-\psi]K\phi$. However, this extension would yield a non-uniform semantics for K and A; a more satisfactory extension would require one to extend the pointed epistemic algebras with becoming-unaware operators. This is not difficult to do once one has extended the framework to deal with the multi-agent case.

Dynamic axioms and rules

+ Gen	From ψ infer $[+\phi](A\psi \rightarrow \psi)$		
+ prop	$[+\phi]p \leftrightarrow p$ for $p \in P$	**– prop**	$[-\phi]p \leftrightarrow p$ for $p \in P$
+ $\wedge$	$[+\phi](\psi \wedge \chi) \leftrightarrow [+\phi]\psi \wedge [+\phi]\chi$	**– $\wedge$**	$[-\phi](\psi \wedge \chi) \leftrightarrow [-\phi]\psi \wedge [-\phi]\chi$
+ $\neg$	$[+\phi]\neg\psi \leftrightarrow \neg[+\phi]\psi$	**– $\neg$**	$[-\phi]\neg\psi \leftrightarrow \neg[-\phi]\psi$
+ A	$(A\chi \rightarrow [+\phi]A\psi) \leftrightarrow ((A\chi \wedge A\phi) \rightarrow A\psi)$	**– A**	$[-\phi]A\psi \leftrightarrow (A\psi \wedge \neg((\neg Ap_1 \wedge \cdots \wedge \neg Ap_n) \rightarrow \neg A\psi))$
+ K	$A\psi \rightarrow ([+\phi]\, K\psi \leftrightarrow K\psi)$	**– K**	$[-\phi]K\psi \leftrightarrow K\psi \wedge [-\phi]A\psi$
$+_{exp}$ **K**	$(A\psi \wedge \neg A\pi \wedge [+\phi]A\pi \wedge \neg(A\pi \rightarrow K\pi)) \rightarrow ([+\phi]K(\psi \vee \pi) \leftrightarrow K\psi)$ for π non-epistemic		

Satisfaction clauses

Narrowing	$((\mathbf{B}, \chi), \Xi) \vDash [-\phi]\psi$	iff	$((\mathbf{B}, \chi), \Xi) - p_1 - \cdots - p_n \vDash \psi$
Expansion	$((\mathbf{B}, \chi), \Xi) \vDash [+\phi]\psi$	iff	$((\mathbf{B}, \chi), \Xi) + p_1 + \cdots + p_n \vDash \psi$
Extension	$((\mathbf{B}, \chi), \Xi), \gamma_{ext} \vDash [+\phi]\psi$	iff	$\gamma_{ext}(((\mathbf{B}, \chi), \Xi), \phi), \gamma_{ext} \vDash \psi$

Fig. 4 Logic of awareness change

In evaluating the intuitive validity of the axioms, it is important to keep in mind that we are axiomatising pure (respectively free) cases of change. Consider for example the axiom for extension (+ K): it merely says that on becoming aware, one does not change one's prior state of knowledge or ignorance with respect to sentences of which one was already aware. This corresponds to the intuitions for cases of pure increase in awareness that were discussed in Section 1.2.1. This is not to say that there are no examples of changes where + K is violated, but only that examples such as these are not necessarily to be considered as counterexamples to the axiom, for they are most likely cases of compound awareness change (involving a change in belief or knowledge as well as awareness), and the axiom is only claimed to apply to pure changes. Similar things can be said for free cases of becoming aware. The difference between free cases and other pure awareness increases resides in a specific axiom, namely $+_{exp}$ K; once again, although $+_{exp}$ K may seem not to hold in a particular example of pure awareness increase, this is most likely because the awareness increase in question is not free.

Remark 2 Note finally that all changes of awareness expressed with the operators $[+\phi]$ and $[-\phi]$ have to be considered to boil down to changes in awareness of the primitive sentences P. This is related to the restrictions placed on interpreted algebras in the static logic (see Remark 1): if changes of awareness were not translatable in terms of changes of awareness of elements of P, then it would be possible to obtain, by applying change operations, interpreted algebras whose interpreting algebras were not generated by elements of P, and the static logic proposed in the previous section would no longer be valid. Hence the limitations imposed by taking the theorist's language as the object language which were discussed in Remark 1 affect not only the ability to represent the static state with the logic, but also the ability to represent changes.

3 Conclusion

Awareness and awareness change are important phenomena for human knowledge and action. Moreover, they pose some interesting logical and philosophical questions. This paper offers a grip on some of these issues. The central aims were to propose a model of awareness change that supports a rigorous understanding of the phenomenon and to use this model to develop logics of awareness change. The first goal was accomplished in Section 1, with the use of the notion of pointed (epistemic) algebras, which represent the agent's epistemic state as understood from his own point of view, and the introduction of operations on these algebras that represent changes of awareness. It was argued that the framework provides both a natural decomposition of the range of ways of becoming aware into two primitive operations, and a principled and clear distinction between awareness change and belief change. The second goal was accomplished in Section 2, where logics of awareness change were developed in the DEL-style framework. Some of the observations made in the development of the DEL-styled logics (in particular, Remarks 1 and 2) argue in favour of the view underlying the aims of this paper: that models and logics of awareness and awareness change are complementary, with the first bringing accuracy, and the second tractability. By proposing both models and logics, and relating one to the other, the current paper offers a complete approach to the problem of awareness change.

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Appendix

A Logic of Awareness Change, AGM Style

What has been done for the DEL-framework can be done for the AGM-framework. In Fig. 5 we present AGM-style postulates for awareness change. The axioms are only given for reflective agents, though similar ones yield analogous results for simple agents. These postulates characterise the change operations defined in Section 1.2.

Theorem 3 *The operation of expansion of pointed epistemic algebras satisfies* (*E 1–8*). *Furthermore, let* $*$ *be any function taking pairs of pointed epistemic algebras and sentences to pointed epistemic algebras, satisfying* (*E 1–8*). *Then, for all* $(\mathbf{B}, \chi)$ *and* ϕ, $(\mathbf{B}, \chi) * \phi = (\mathbf{B}, \chi) + \phi$.

In the following postulates, $(\mathbf{B}, \chi)$ is a pointed epistemic algebra, ϕ is a sentence, we write $(\mathbf{B}+\phi, \chi')$ for $(\mathbf{B}, \chi)+\phi$ and $(\mathbf{B}-\phi, \chi')$ for $(\mathbf{B}, \chi)-\phi$. $f(\phi)$ in (N 2) and (N 4) is as in Definition 7.

(E 1)	$\mathbf{B}+\phi$ is an interpreted (epistemic) algebra
(E 2)	$\phi \in \mathbf{B}+\phi$ (Success)
(E 3)	if $\phi \in \mathbf{B}$, $\mathbf{B} = \mathbf{B}+\phi$ (Vacuity)
(E 4)	for a sentence ψ, $\psi \notin B_{I\cup\{\phi\}}$, then $\psi \notin \mathbf{B}+\phi$ (Minimality)
(E 5)	there is an embedding from $\mathbf{B}$ into $\mathbf{B}+\phi$ (Inclusion)
(E 6)	for all $\psi \in \mathbf{B}+\phi \setminus \mathbf{B}$ and $\chi \in \mathbf{B}$, if $\psi \Rightarrow_{\mathbf{B}+\phi} \chi$ (respectively, $\chi \Rightarrow_{\mathbf{B}+\phi} \psi$), then $\psi = \psi_1 \wedge \psi_2$ (resp. $\psi = \psi_1 \vee \psi_2$) with $\psi_1 \in \mathbf{B}$, $\psi_2 \in \mathbf{B}+\psi \setminus \mathbf{B}$ and $\psi_1 \Rightarrow_{\mathbf{B}} \chi$ (resp. $\chi \Rightarrow_{\mathbf{B}} \psi_1$). (Freeness)
(E 7)	For all $\psi \in \mathbf{B}$, $\chi' \Rightarrow_{\mathbf{B}+\phi} \psi$ iff $\chi \Rightarrow_{\mathbf{B}} \psi$ (Conservativity)
(E 8)	For all non-epistemic $\psi \in \mathbf{B}+\phi \setminus \mathbf{B}$ such that $\chi \not\Rightarrow_{\mathbf{B}+\phi} \psi$, $\chi' \Rightarrow_{\mathbf{B}+\phi} \neg K\psi$ (Ignorance)
(N 1)	$\mathbf{B}-\phi$ is an interpreted (epistemic) algebra
(N 2)	if $\psi \in B_{f(\phi)}$, then $\psi \notin \mathbf{B}-\phi$ (Success)
(N 3)	if $\phi \notin \mathbf{B}$, $\mathbf{B} = \mathbf{B}-\phi$ (Vacuity)
(N 4)	for $f(\phi) = \{\pi_1, \ldots, \pi_n\}$, $\mathbf{B}$ is an extension of $\mathbf{B}-\phi$ by $\pi_1, \ldots, \pi_n$ (Recovery)
(N 5)	For all $\psi \in \mathbf{B}-\phi$, $\chi' \Rightarrow_{\mathbf{B}-\phi} \psi$ iff $\chi \Rightarrow_{\mathbf{B}} \psi$ (Conservativity)

Fig. 5 AGM-style postulates

Extensions of pointed epistemic algebras satisfy (*E 1–5*), (*E 7*), *(E 8). Furthermore, let* $*$ *be any relation between pairs of pointed epistemic algebras and sentences, and pointed epistemic algebras, satisfying* (*E 1–5*), (*E 7*), (*E 8*). *Then, for all* $(\mathbf{B}, \chi)$ *and* ϕ, $(\mathbf{B}, \chi) * \phi$ *is an extension of* $(\mathbf{B}, \chi)$.

The operation of narrowing of pointed epistemic algebras satisfies (*N 1–5*). *Furthermore, let* $*$ *be any function taking pairs of pointed epistemic algebras and sentences to pointed epistemic algebras, satisfying* (*N 1–5*). *Then, for all* $\mathbf{B}$ *and* ϕ, $(\mathbf{B}, \chi) * \phi = (\mathbf{B}, \chi) - \phi$.

B Proofs

As noted in Section 1.1, we assume here that the algebras are finite. Although several of the proofs given below will use this assumption, it is generally stronger than required, and can often be weakened. The following notational convention shall be adopted (see also footnote 13): in the case of mappings between algebras, the same symbol shall be used both for an element of original algebra and its image under the mapping.

Fact 1 *For pointed* (*epistemic*) *algebras* $((B_I, B, q), \chi)$ *and* $((B_{I'}, B', q'), \chi')$, *consider the following conditions.* (1) $I \subseteq I'$. (2) *for all* $\phi, \psi \in ((B_I, B, q), \chi)$, $\phi \Rightarrow_{(B_{I'}, B', q')} \psi$ *iff* $\phi \Rightarrow_{(B_I, B, q)} \psi$. (3) *for all* $\phi \in (B_{I'}, B', q')$, $\chi' \Rightarrow_{(B_{I'}, B', q')} \phi$ *iff* (*a*) $\chi \Rightarrow_{(B_{I'}, B', q')} \phi$ *or* (*for epistemic algebras*) (*b*) $\phi = \neg K\psi$ *where* $\psi \in (B_{I'}, B', q') \setminus (B_I, B, q)$ *is non-epistemic and* $\chi \not\Rightarrow_{(B_{I'}, B', q')} \psi$ *or* (*c*) ϕ *is a*

$\Rightarrow_{(B_{I'},B',q')}$*-consequence of a set of* ϕ_i *each satisfying one of* (a) *or* (b). (1)–(3) *are satisfied iff there is an embedding of* $((B_I, B, q), \chi)$ *into* $((B_{I'}, B', q'), \chi')$

Proof The right to left direction is straightforward. For the left to right direction, let σ_i be the embedding of $B_{I'}$ into B_I generated by $I \subseteq I'$, and $\sigma_b : B \to B'$ be defined as follows: for all $\phi \in B_I$, $\sigma_b(q(\phi)) = q'(\phi)$. (1) and (2) ensure that these are well defined (epistemic) homomorphisms; they yield an embedding of (B_I, B, q) into $(B_{I'}, B', q')$. (3) ensures that this provides an embedding of the pointed algebras. □

Remark 3 The conditions (1–3) are formalised and slightly strengthened versions of the properties of pure cases of becoming aware stated in Section 1.2.1. On the one hand (1) states not only that new sentences come into play, but also the sentences which initially counted as primitive (the elements of I) count as primitive after the change. On the other hand (3) renders the condition on knowledge in the case of reflective agents more precise: the agent does not know any new non-epistemic sentence which is not implied by his prior knowledge (according to the posterior consequence relation), he only knows of his ignorance of such sentences (and of the consequences of this ignorance).

Proof of Proposition 1 Let $\phi_1, \ldots, \phi_n$ be the sequence of elements in $I_2 \setminus I_1$, and consider $\mathbf{B}_+ = \mathbf{B_1} + \phi_1 + \cdots + \phi_n$. It follows immediately from Definitions 4 and 5 that B_{I_+} and B_{I_2} are isomorphic. The isomorphism between B_{I_+} and B_{I_2} and the injective homomorphism σ_B from B_1 to B_2 generate a homomorphism σ_+ from B_+ to B_2 which is injective on $B_1 \subseteq B^+$. Let y be the maximal element of $\sigma_+^{-1}(\bot)$ and pick $\psi \in \mathbf{B}_+$ with $q_+(\psi) = y$. By construction, $\mathbf{B}_+/\psi$ is isomorphic to $\mathbf{B_2}$. (This also applies in the case of epistemic algebras, since σ_+ is an epistemic homomorphism, so $\{x \in B_+ | \ y \leq x\}$ is a modal filter.) It is straightforward to check that the isomorphism between interpreted (epistemic) algebras preserves the knowledge elements. □

Proof of Theorem 1 Soundness is straightforward. For completeness, consider a $\mathbf{Aw^r}$-consistent set of sentences of $\mathcal{L}_P^{KA}$, Σ. Using traditional methods ("Lindenbaum's Lemma"), extend it to a maximal consistent set Σ^m. Take Ξ to be the set of propositional elements of Σ^m; since the latter is maximally consistent, the former is as well. Let $I = \{p \in P \mid Ap \in \Sigma^m\}$, and $\mathbf{B} = (B_I, B, q)$, where B and B_I are epistemic algebras, B is isomorphic to B_I and q is the isomorphism. A1–A4 guarantee that, for any $\phi \in \mathcal{L}_P^{KA}$, $A\phi \in \Sigma^m$ iff $\phi \in \mathbf{B}$; A5 guarantees that one can take $A\phi \Leftrightarrow_{\mathbf{B}} \top$ for all $\phi \in \mathbf{B}$. Take χ to be the minimal element of $\{\psi \mid K\psi \in \Sigma^m\}$; A0 guarantees that this is an element of $\mathbf{B}$, 4 guarantees that $\chi \Rightarrow_{\mathbf{B}} K\chi$, 5 guarantees that, for all $\psi \in \mathbf{B}$, if $\chi \not\Rightarrow_{\mathbf{B}} \psi$, then $\chi \Rightarrow_{\mathbf{B}} \neg K\psi$, and the consistency of Σ^m combined with the coincidence between the consequence relations in $\mathbf{B}$ and $\mathcal{L}_P^{KA}$ guarantee that $\chi \not\Leftrightarrow_{\mathbf{B}} \bot$. Finally, T ensures that, for any $\psi \in \mathcal{L}_P$, if $\chi \Rightarrow_{\mathbf{B}} \psi$ then $\psi \in \Xi$. So $((\mathbf{B}, \chi), \Xi)$ is a pair, consisting of an epistemic interpreted algebra and a maximal consistent set of propositional sentences, that satisfies Σ^m and thus Σ. □

Proof of Theorem 2 Soundness is straightforward. For completeness, consider Σ, a consistent set of sentences $\mathcal{L}_P^{KA+-}$. Take a maximal consistent extension Σ^m, and construct $((\mathbf{B}, \chi), \Xi)$ using the $\mathcal{L}_P^{KA}$-fragment of Σ^m as in the proof of Theorem 1.

For expansion and narrowing: using the set $\{\psi | [+\phi]\psi \in \Sigma^m\}$ (resp. $\{\psi \ | [-\phi]\psi \in \Sigma^m\}$), and the same technique as in the proof of Theorem 1, construct pairs of interpreted epistemic algebras and maximal consistent sets of propositional letters, $((\mathbf{B}_\phi, \chi_\phi), \Xi_\phi)$, for each $\phi \in \mathcal{L}_P^{KA}$. It remains to show that, for any ϕ, the pair constructed is the result of applying the appropriate expansion (resp. narrowing) operations. We consider just expansion; the case of narrowing is similar. $+$ prop, $+ \wedge$ and $+ \neg$ guarantee that $\Xi_\phi = \Xi$. It follows from $+$ A and A1–A4 that $\mathbf{B}_\phi = \mathbf{B} + p_1 + \cdots + p_n$, for $f(\phi) = \{p_1, \ldots, p_n\}$. $+$ K implies that $\chi_\phi \Rightarrow_{\mathbf{B}_\phi} \chi$ and that for any $\psi \in \mathbf{B}$ such that $\chi \not\Rightarrow_{\mathbf{B}} \psi$, $\chi_\phi \not\Rightarrow_{\mathbf{B}_\phi} \psi$. $+_{exp}$ K implies that, for each $\psi \in \mathbf{B}$ such that $\chi \not\Rightarrow_{\mathbf{B}} \psi$, and for any non-trivial Boolean combination π of elements in $f(\phi)$ but not in $\mathbf{B}$, $\chi_\phi \not\Rightarrow_{\mathbf{B}_\phi} \psi \vee \pi$, and so, by 5, $\chi_\phi \Rightarrow_{\mathbf{B}_\phi} \neg K(\psi \vee \pi)$. Hence χ_ϕ is the conjunction of χ with all such $\neg K(\psi \vee \pi)$, and thus $((\mathbf{B}_\phi, \chi_\phi), \Xi_\phi) = ((\mathbf{B}, \chi), \Xi) + p_1 + \cdots + p_n$.

Now consider extension: For any ϕ, construct a pair $((\mathbf{B}_\phi, \chi_\phi), \Xi_\phi)$ consisting of an interpreted epistemic algebra and a maximally consistent set of propositional letters as follows. Take Ξ_ϕ to be the set of propositional elements ψ such that $[+\phi]\psi \in \Sigma^m$; since the latter is maximally consistent, the former is as well. Let $I_\phi = \{p \in P \mid [+\phi]Ap \in \Sigma^m\}$, and π be the supremum of $\{\psi \in B_{I_\phi} \setminus B_I | [+\phi]K\neg\psi \wedge \neg A\psi \in \Sigma^m\}$. Let $\mathbf{B}_\phi = (B_{I_\phi}, B_\phi, q_\phi)$, where B_{I_ϕ} is the epistemic algebra generated by I_ϕ and B_ϕ is the quotient by π, with q_ϕ the quotient map. (It is straightforward to check that this is a well-defined epistemic quotient.) Finally, let χ_ϕ be the minimal element of $\{\psi | [+\phi]K\psi \in \Sigma^m\}$. By the reasoning in the proof of Theorem 1, this is a well-defined knowledge element. By $+$ prop, $+ \wedge$ and $+ \neg$, $\Xi_\phi = \Xi$. It follows from $+$ A and A1–A4 and the construction of B_ϕ that $\mathbf{B}_\phi = \mathbf{B} + p_1 + \cdots + p_n/\pi$ (where $f(\phi) = \{p_1, \ldots, p_n\}$). Finally, let χ' be the image of χ in $\mathbf{B}_\phi$. It follows from $+$ K that $\chi_\phi \Rightarrow_{\mathbf{B}_\phi} \chi'$ and that for any $\psi \in \mathbf{B}$ such that $\chi \not\Rightarrow_{\mathbf{B}} \psi$, $\chi_\phi \not\Rightarrow_{\mathbf{B}_\phi} \psi$. However, by construction, for any $\psi \in \mathbf{B}_\phi \setminus \mathbf{B}$ such that $\psi \not\Leftrightarrow_{\mathbf{B}_\phi} \top$, $\chi_\phi \not\Rightarrow_{\mathbf{B}_\phi} \psi$. Thus $(\mathbf{B}_\phi, \chi_\phi)$ is an extension of $(\mathbf{B}, \chi)$ by $p_1, \ldots, p_n$ and Σ is satisfied in $((\mathbf{B}, \chi), \Xi), \gamma_{ext})$ where $\gamma_{ext}(((\mathbf{B}, \chi), \Xi), \phi) = ((\mathbf{B}_\phi, \chi_\phi), \Xi)$. □

Proof of Theorem 3 It is straightforward to check that the operations satisfy the appropriate postulates.

Consider the other direction, first of all for the cases of expansion and extension. Let $*$ be an operation on pointed epistemic algebras satisfying (E 1–5), (E 7), (E 8) and consider an arbitrary algebra $(\mathbf{B}, \chi)$ and an arbitrary sentence ϕ. Let $(\mathbf{B}', \chi') = (\mathbf{B}, \chi) * \phi$. If $\phi \in \mathbf{B}$, by (E 3), $\mathbf{B}' = \mathbf{B} = \mathbf{B} + \phi$. Consider now the case where $\phi \notin \mathbf{B}$. By (E 5), $I \subseteq I'$; by (E 2), $\phi \in B_{I'}$; so $B_{I \cup \{\phi\}}$ is a subalgebra of $B_{I'}$. However, by (E 4), $B_{I'}$ is a subalgebra of $B_{I \cup \{\phi\}}$; so $B_{I'} = B_{I \cup \{\phi\}}$. By (E 5) again, there is an embedding σ_B of B into B'. So $\mathbf{B}'$ is an extension of $\mathbf{B}$ by ϕ. Suppose that, in addition, (E 6) holds: then, for any $\psi \in \mathbf{B} + \phi \setminus \mathbf{B}$, if $\psi \Leftrightarrow_{\mathbf{B}+\phi} \bot$, then $\psi = \bot$. Hence, $B' = B \otimes B_{\{q'(\phi)\}}$, and so

$\mathbf{B}' = \mathbf{B} + \phi$. Moreover, (E 7) implies that $\chi' \Rightarrow_{\mathbf{B}'} \chi$ and that for any $\psi \in \mathbf{B}$ such that $\chi \not\Rightarrow_{\mathbf{B}} \psi$, $\chi' \not\Rightarrow_{\mathbf{B}'} \psi$. (E 8) implies that, for each non-epistemic $\psi \in \mathbf{B}' \setminus \mathbf{B}$ such that $\chi \not\Rightarrow_{\mathbf{B}'} \psi$, $\chi' \Rightarrow_{\mathbf{B}'} \neg K\psi$. This determines the value of χ' in $\mathbf{B}'$: it is the conjunction of χ with the $\neg K\psi$ for all non-epistemic $\psi \in \mathbf{B}' \setminus \mathbf{B}$ such that $\chi \not\Rightarrow_{\mathbf{B}'} \psi$. So $(\mathbf{B}, \chi) * \phi$ is an extension of $(\mathbf{B}, \chi)$ by ϕ, and it is an expansion if (E 6) is satisfied.

Now consider the case of narrowing. Let $*$ be an operation on pointed (epistemic) algebras satisfying (N 1–5) and consider an arbitrary algebra $(\mathbf{B}, \chi)$ and an arbitrary sentence ϕ. Let $(\mathbf{B}', \chi') = (\mathbf{B}, \chi) * \phi$. If $\phi \notin \mathbf{B}$, by (N 3), $\mathbf{B}' = \mathbf{B} = \mathbf{B} - \phi$. Consider now the case where $\phi \in \mathbf{B}$. By (N 4), $\mathbf{B} = \mathbf{B}' + \pi_1 + \cdots + \pi_n/\psi$ for $f(\phi) = \{\pi_1, \ldots \pi_n\}$ and $\psi \in (\mathbf{B}' + \pi_1 + \cdots + \pi_n) \setminus \mathbf{B}'$. By Definitions 5 and 6, $I = I' + \{\pi_1, \ldots, \pi_n\}$; by (N 2), $\pi_i \notin B_{I'}$ for all i; so $I' = I \setminus \{\pi_1, \ldots, \pi_n\}$ and thus $B_{I'} = B_{I \setminus f(\phi)}$. By (N 4), q' is the restriction of q to $B_{I \setminus f(\phi)}$, and B' is isomorphic to the image of $B_{I \setminus f(\phi)}$ in B under q. Hence $\mathbf{B}' = \mathbf{B} - \phi$. Finally, by (N 5), χ' is the minimal element of $\{\psi \in \mathbf{B}' \mid \chi \Rightarrow_{\mathbf{B}} \psi\}$. Thus $(\mathbf{B}, \chi) * \phi = (\mathbf{B}, \chi) - \phi$. □

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